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Transverse waves in a stratified fluid: An educational experiment

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Summary. — An object in equilibrium within a stratified fluid begins to oscillate when displaced and generates internal gravity waves, which are transverse waves despite the fact that they develop in bulk. Although these waves are extremely difficult to observe, they are typical of natural environments such as the atmosphere and the deep ocean. We propose an educational experiment that allows to understand how gravity waves are generated, how they propagate in stratified fluids, and to appreciate their presence through the effects on other objects inside the fluid.

1. – Introduction

Internal gravity waves occur in stratified fluids, that is, in fluids whose density varies with depth, such as in the ocean or the atmosphere. Unlike surface waves that propagate along the interface between two immiscible fluids of different densities, internal waves move within the fluid itself, oscillating between internal layers of the fluid. These waves arise when a fluid parcel displaced vertically experiences a restoring force due to gravity and buoyancy, generating a vertical oscillation of the volume around its equilibrium position. The frequency of these oscillations is known as the Brunt-Väisälä frequency [1-4] and depends on the amplitude of the density gradient parallel to the gravity force. Internal waves play a crucial role in the transfer of energy within stratified fluids, influencing large-scale oceanic circulation, as well as atmospheric circulation [3]. In the ocean, they contribute to mixing, affecting nutrient transport and global climate dynamics. In the atmosphere, they affect weather patterns and turbulence, influencing jet streams and storm development. These waves also appear in astrophysics, for example in the phenomena governing the stellar interiors, where they affect energy transport and the evolution of the stars [5]. Internal gravity waves also occur at the mesoscale in stratified fluids at rest, where spontaneous density fluctuations are overstabilised by the presence of the density

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gradient and the gravitational field. In this condition, long-wavelength non-equilibrium fluctuations at the millimeter scale oscillate vertically within the fluid bulk, generating propagating modes that transport energy. This behavior has been described theoretically and experimentally for binary and ternary mixtures [6-8].

Despite their importance, internal gravity waves are challenging to study because they occur in transparent fluids and are difficult to detect, as they primarily involve changes in the refractive index rather than the shape of the surface of the fluid. Experimental studies often rely on optical techniques such as schlieren, shadowgraph [9], or particle image velocimetry to visualize the wave dynamics. Numerical simulations provide further insight into their behavior in complex environments, including interactions with topography, non-linear effects, and interaction with levitating objects [10]. Understanding internal gravity waves is essential in a broad range of scientific disciplines for improving climate models, predicting ocean mixing, and studying mesoscale fluid dynamics in planetary atmospheres.

Teaching concepts related to internal gravity waves is therefore of great importance to prepare the scientists of tomorrow, but related subjects are challenging and require innovative methods.

In this paper, we present educational experiments that can be used to investigate quantitatively internal gravity waves in a stratified fluid. We show how internal gravity waves can be generated using a floating object and detected through their impact on its dynamics, including the damping of its oscillations and their revival.

This work expands our previous publications [4,11], where the same experiments were discussed for their effectiveness in conveying fundamental concepts about pressure and fluid dynamics, mainly for teaching activities for high school and undergraduate students. Here, we focus on a more advanced understanding of the interplay between waves and oscillations in a stratified fluid. In particular, we focus on the effect of the internal gravity waves on the oscillatory motion of a body immersed in a stratified fluid. In our previous works [4,11], we interpreted the damping and the following restarting of the oscillations as due to the beatings between the object motion and the internal gravity wave. This explanation turned out to be not completely exhaustive. In this work, we describe in detail the subtle mechanism of interaction that gives rise to this engaging behaviour. We think that the explanation is elegant and not trivial; therefore, it is particularly interesting to be discussed with both high school and undergraduate students.

The effect we discuss was observed in two different experiments already published [4, 11], but the interpretation we present is completely new. To have a self-consistent paper, however, in the next paragraph, we report a description and the most significant educational aspects of the two experiments. Then we present the basic equations that describe the oscillations of a body in a stratified medium and the basic formulas, such as the expression for the Brunt-Väisälä frequency. Readers interested in a more detailed treatment of this topic are referred to refs. [4,11]. In the following section, we discuss in detail how internal gravity waves originate and how they interact with the oscillating body that has generated them, giving rise to its peculiar behaviour.

2. – Experiment and results

In previous papers, we presented two experiments suitable for familiarizing students with the physics of fluids, including the basic principles of hydrostatics, such as the Stevino law and Archimedes and Pascal principles, but also for introducing them to the properties of stratified fluids [4, 11]. In both experiments, an object is in almost stable equilibrium at a certain height in a container filled with water, where a layer of coarse salt deposited on the bottom is diffusing and creates a density profile (see fig. 1).

In the first case [4], the object is a cork weighted by putting inside it a steel ball and sealing it back, so that its density is slightly larger than that of water. During the salt dissolution, the cork is located in the layer where its density is the same as its own, so that at any time it is in almost stable equilibrium. This experiment is reminiscent of one described by Galileo Galilei [12] in the "Dialogues Concerning Two New Sciences". Here Sagredo tells about a trick proposed to some friends where a wax ball thrown in water appeared to stay still in the middle of the container: *"In the bottom of a vessel I placed some salt water and upon this some fresh water; then I showed them that the ball [of wax] stopped in the middle of the water, and that, when pushed to the bottom or lifted to the top, would not remain in either of these places but would return to the middle"*.

A similar experiment can be performed with a slightly different scheme, namely a variant of a Cartesian diver apparatus [13, 14]. A hollow object with a bottom opening that allows water to pass through and containing a small air bubble floats in a stratified liquid at the height where its density is matched. The volume of the air bubble can be altered by changing the pressure on top of the sealed container, and the equilibrium position of the diver can be varied [11]. In both cases, when the floating body is pushed off its equilibrium position, it starts oscillating driven by a restoring force. Its frequency is known as Brunt-Väisälä frequency and, as will be detailed in the next paragraph, is strictly related to the density gradient.

The motion of the objects can be filmed after they have been displaced from their equilibrium position, and the oscillations can be monitored using a tracking program [15]. By studying how the oscillation frequency of a fixed-density object, such as the cork, varies in time, it is possible to study the evolution of the density gradient around a layer of fluid of a given density [4]. Conversely, when a Cartesian diver is used, by varying its

Fig. 1. – A weighted cork (left image) and a cartesian diver (right image) are suspended in a stratified liquid in a cylindrical container. In the case of the diver, the cylinder is sealed on top with a membrane that can be deformed when compressed and transfers the applied pressure to the fluid.

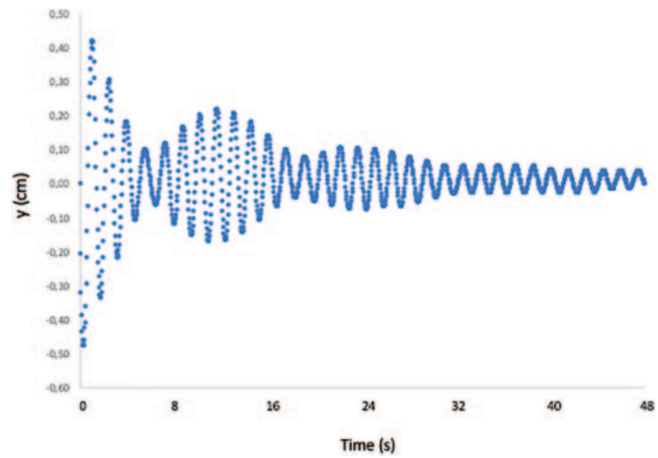

Fig. 2. – The oscillations of a weighted cork immersed in a stratified fluid measured using a tracking program [15]. Initially, the oscillations are damped, then they alternate phases where they increase and reduce without any further manipulation of the system.

density through the application of an external pressure, it is possible to map at the same time the density profile at various heights [11].

The primary interest in conveying the fundamental concepts about pressure and fluid dynamics could be to characterize uniquely the frequency of oscillation. However, other relevant features are present that are worth noticing. As can be observed in fig. 2, oscillations are naturally damped, intuitively due to the viscous dissipation inside the fluid, but they are periodically amplified again after a certain time and exhibit a behavior similar to that observed in the presence of beats, which occur when two waves with similar frequencies interfere.

In an attempt to understand the complexity of this phenomenon, students are naturally led to find answers to fundamental questions about the system, or can be guided to do so by the teacher.

Basically we are interested in answering the following questions:

- Why does the body oscillate?
- Why are the oscillations damped?
- Why, after stopping, the oscillations start again?

3. – The motion of a body in a stratified fluid

In this section we will recall the basic concepts of harmonic oscillations of a body, analyse the causes that lead to a damping of the oscillations and investigate the source of the revival of the oscillations. We will take the point of view adopted by the 17 year old Enrico Fermi in his essay "Caratteri distintivi dei suoni e loro cause" for the admission to the Scuola Normale Superiore di Pisa in 1918. This choice is particularly appropriate because, at the time he took this exam, Enrico Fermi was about the same age as the students we are addressing with our experiment, which may lead them to view the text with greater interest. Moreover, the approach Fermi used to tackle the problem of describing the distinctive features of sounds and their causes is highly analytical, easy to explain to students, and contains the key questions we need to answer in order to describe oscillations in a stratified medium.

The two main questions addressed by Enrico Fermi that we want to address to the students are i) How do bodies vibrate? ii) How does the medium surrounding them transmit these vibrations?

3·1. *Brunt-Väisälä oscillations.* – A body immersed in a liquid exhibits three main buoyancy regimes, depending on the relation between its density ρ_0 and that of the liquid ρ. When $\rho_0 < \rho$ the body reaches the surface of the liquid where it floats, while for $\rho_0 > \rho$ it sinks to the bottom of the liquid. In the peculiar case where $\rho_0 = \rho$, the body is neutrally buoyant and can remain still indefinitely at any depth inside the liquid.

The situation changes drastically when the body is immersed in a fluid with a stable density stratification $\rho(z)$, where z is the vertical axis pointing upwards. A stable stratification is achieved when the density of the fluid increases with depth [16], that is, when $\partial\rho/\partial z < 0$. If we assume that the density of the body lies between the densities of the top and bottom layers of the liquid, then the body moves vertically until it finds a layer of fluid matching its density. In this layer, the body is in a stable equilibrium condition, where the weight force is exactly balanced by the buoyancy force. For simplicity, in the following we will assume that this equilibrium condition is at $z = 0$ and that the density in the neighborhood of this equilibrium position changes linearly according to the equation $\rho(z) = \rho_0 + \frac{\partial\rho}{\partial z} z$. When the body is displaced by an amount z from its equilibrium position it will feel a restoring force that can be written as

$$F_r = gV_0 \frac{\partial\rho}{\partial z} z, \tag{1}$$

where V_0 is the volume of the body. Notice that we have previously assumed that $\partial\rho/\partial z < 0$, and eq. (1) is perfectly equivalent to Hooke's law for an elastic force, where the quantity $gV_0\partial\rho/\partial z$ plays the role of an elastic constant. According to the second principle of Newtonian dynamics $\rho_0 V_0 \ddot{z} = F_r$, we obtain the equation of motion $\ddot{z} + \omega_0^2 z = 0$ of the body, where

$$\omega_0 = \sqrt{-\frac{g}{\rho_0}\frac{\partial\rho}{\partial z}} \tag{2}$$

is the Brunt-Väisälä angular frequency that characterises the oscillations in a stratified fluid [3].

If we assume that the initial displacement of the body from its equilibrium position is $z(0) = A$ and its initial velocity $\dot{z}(0) = 0$, then the solution of the equation of motion is a simple harmonic oscillation in the form $z(t) = A\cos$.

4. – The subtle interplay between waves and oscillations in a stratified fluid

The simple model depicted in the previous section does not take into account the gradual loss of energy of the body determined by its interaction with the surrounding medium. The traditional textbook approach to justify the damping of harmonic oscillations is to add to the forces acting on the body a frictional viscous force $F_v = -2\gamma\rho_0 V_0 \dot{z}$

proportional to its velocity and opposing its motion, where γ is a damping constant. With the inclusion of this force, the equation of motion becomes

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = 0. \tag{3}$$

This second-order differential equation can be solved with the traditional methods used for the solution of homogeneous linear differential equations. Assuming that the initial conditions are again $z(0) = A$ and $\dot{z}(0) = 0$, and that $\gamma < \omega_0$, the solution of the equation is

$$z(t) = A\exp(-\gamma\, t)\cos(\omega_1\, t + \phi), \tag{4}$$

where ϕ is a phase constant and $\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$ is the angular frequency of the oscillations. Since $\omega_1 < \omega_0$, the damping determines a slowdown of the oscillations.

4·1. *Emission of internal gravity waves.* – If we assume that the viscous damping determined by the liquid surrounding the body is negligible, one could argue that the body should oscillate indefinitely in a regime of undamped harmonic oscillations. However, it turns out that this is not the case, because another very important effect is at play in determining the loss of energy of the oscillator, namely the emission of radiation in the form of internal gravity waves. Indeed, since the liquid is incompressible, a vertical movement of the body determines a vertical displacement of an equivalent volume of liquid in the opposite direction. Therefore, the harmonic oscillation of the body determines a harmonic motion of the liquid surrounding it, which in turn triggers a harmonic motion of the adjacent fluid through the cascade process typical of wave propagation. Quite interestingly, it can be shown that the loss of energy of the oscillator determined by the emission of a wave can be written exactly in the form of eq. (3) and, correspondingly, that its motion obeys eq. (4). If we call $z_{ig}(x,t)$ the vertical displacement of the fluid at a horizontal distance x from the body at time t, this represents the wave function describing the internal gravity waves inside the stratified liquid. Here for simplicity, we are restricting to one horizontal coordinate x and neglecting the dependence on the other linear coordinate y. This is the condition met in the experiments in the narrow container like the one shown in fig. 3. Since the fluid oscillates with the same frequency as the body and its vertical displacement is the opposite of that of the body at $x = 0$, we can write directly the wave function that describes the internal gravity waves by propagating in the horizontal direction the perturbation described by eq. (4),

$$z_{ig}(x,t) = -A\exp\left[-\frac{\gamma}{\omega_1}\,(k\,x - \omega_1 t)\right]\cos(kx - \omega_1\, t + \phi), \tag{5}$$

where k is the wave number of the internal gravity wave.

The process of emission of an internal gravity wave determined by the oscillations of the body is depicted schematically in fig. 4(a), where for the sake of simplicity we have represented a progressive and a regressive wave induced by the oscillation of the body, while in a geometry like the one adopted in the experiments in a cylindrical container, like the one shown in fig. 1, the oscillations of the body determine the emission of a divergent wave. Initially, the amplitude of the oscillations is large, and this results in a large amplitude of the forefront of the internal gravity wave. The damping of the oscillations determined by the transfer of energy to the wave gives rise to a gradual decrease in time of the amplitude of the wave, until eventually the oscillations of the body stop.

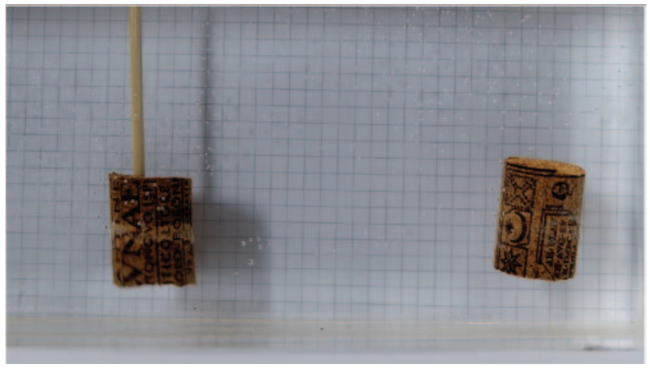

Fig. 3. – Two corks are placed in a narrow tank. The left one is pushed down with a stick and triggers the emission of an internal gravity wave that, when it reaches the second one, drives its oscillation.

4·2. *Driven oscillations.* – An internal gravity wave propagating in a stratified liquid can trigger the motion of a body initially at rest in its equilibrium position. This is the case of the experiment shown in fig. 3, and discussed in ref. [4], where the oscillation of the first body triggers the emission of an internal gravity wave, which propagates in the horizontal direction until it reaches the second body and drives its oscillations. Remarkably, the same process can also occur when a single body is present. In this case, the oscillations of the body determine the emission of an internal gravity wave (fig. 4(a)). This wave propagates away from the body until it reaches the lateral walls of the container hosting the stratified liquid. Here it gets reflected by the wall and heads back towards the body that originated it and drives its oscillations (fig. 4(b)). The phase of the wave

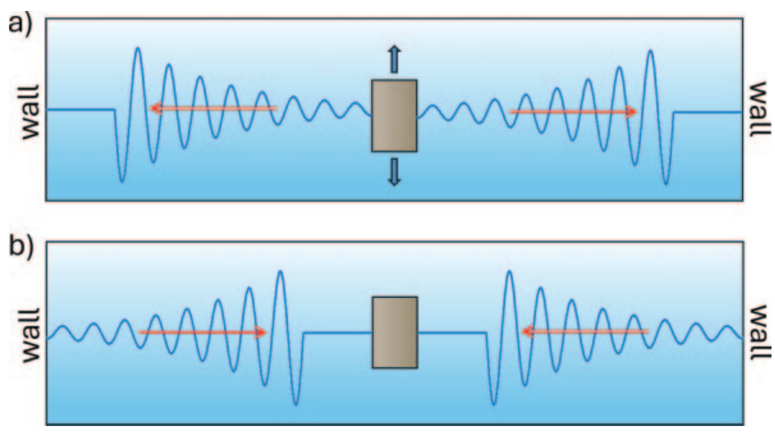

Fig. 4. – (a) The oscillating body loses energy due to the emission of an internal gravity wave that propagates through the stratified liquid. The loss of energy determines a damping of the oscillation and, in turn, the gradual decrease of the amplitude of the internal gravity wave. The loss of energy eventually stops the oscillations. (b) In the presence of lateral bounds, the internal gravity wave is reflected by the walls and inverts its direction of propagation, heading towards the still body. When it reaches the body, it transfers energy to it and determines the start of forced oscillations.

when it reaches the body depends critically on the length of the path traveled, which, under generic conditions, will not be an integer multiple of the wavelength of the wave. For this reason, if the body is still oscillating when the reflected wave starts driving its oscillations, the oscillations of the body will not be in phase with those of the driving force. Consequently, the oscillations will undergo a phase shift, which is apparent in the experimental results shown in fig. 2.

5. – Conclusions

We recently proposed two different experiments able to catch the attention of students and to convey many concepts about the physics of fluids [4, 11]. Both experiments have been developed during bachelor degree thesis and tested in high schools. They are extremely effective in catching students' interest and can be easily proposed to students by incorporating them into an inquiry-based learning path, also taking advantage of the numerous connections with related aspects such as natural phenomena, historical aspects, or highly relevant research topics [4, 11]. In this work, we have shown how the physical understanding emerging from these same two experiments allows to introduce and explore more deeply the topic of gravity waves in connection to Brunt-Väisälä oscillations, and have shown how they can be explained with simple arguments. This approach can be more generally used to introduce the transport and transfer of energy by a wave. Internal gravity waves exhibit several interesting and non-trivial aspects and are directly linked to natural phenomena, but they are difficult to detect because, as mentioned, they primarily involve variations in the refractive index. These experiments allow us to bypass this issue by understanding and modeling gravity waves through their effects on an oscillating body and not through a direct visualization.

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