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# Different distinguishability quantifiers for quantum non-Markovianity

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**Summary.** — In this note, the definitions of quantum non-Markovianity based on the monotonicity of distinguishability quantifiers between two evolving quantum states are discussed. In particular, we elaborate on going beyond the commonly used trace distance and using entropic quantifiers.

## 1. – Evolution of open quantum system from the global perspective

Quantum technologies change from the auspicious vision of the future into a development with more and more relevance and applications. That is why a description of realistic quantum systems, with an applicability for a wide range of physical parameters and affected by external noise, is gaining significance. With this the mathematical description of open, *i.e.*, influenced by some external degrees of freedom, quantum systems is evermore demanding as many of the prevalent assumptions about the evolution or system itself are not accurate anymore. New phenomena, one of which is quantum non-Markovianity, come into play.

The description of open quantum system dynamics often takes place in terms of evolution equations for the reduced density operator $\rho(t) = \mathrm{Tr}_E(\rho_{SE}(t))$ (where $t$ is time, $\mathrm{Tr}_E(\cdot)$ is the partial trace with respect to the environmental degrees of freedom, which are at time $t = 0$ uncorrelated with the reduced system, and $\rho_{SE}(t)$ is the total density matrix undergoing unitary evolution), called master equations [1]. The reduced density operator contains complete information about the outputs of all single-time "local experiments" and in this sense the description contains the minimal necessary knowledge about the environmental degrees of freedom to describe open quantum system dynamics. Accordingly, the division we make between the reduced system and the environment can be motivated by the fact that the first one includes the degrees of freedom which are of interest and the latter is thought as a "disturbance". This point of view should, however, be taken with a certain care. For instance, the quantum non-Markovianity, often connected with more profound influence of the environment on the reduced systems, can be beneficial, *e.g.*, for error correction schemes [2] and quantum teleportation [3]. Additionally, the knowledge of some additional properties of the environment can be helpful for better understanding some properties of the reduced dynamics, as we will see on an example of quantum non-Markovianity.

Markovianity is a well-defined feature of a classical stochastic process which is memoryless, *i.e.*, which future is only influenced by its current value and not by the past ones. However, due to the invasive nature of measurement in quantum physics, the analogous to the classical definition of non-Markovianity with measurement probabilities is not straightforward and many different, nonequivalent definitions of quantum non-Markovianity exist [4, 5]. These definitions mostly see quantum non-Markovianity as a property of the quantum dynamical map $\Lambda_t : \rho(t) = \Lambda_t[\rho(0)]$ itself (*i.e.*, no knowledge about the underlying microscopical realization of the evolution on the level of the total state is needed); there are, however, some exceptions [5]. The external degrees of freedom are nonetheless essential for occurrence of quantum memory effects as the closed, unitary evolution is always free from them. As an example, the BLP condition can be connected with the creation of system-environmental correlations and changes in the environment [6]. The dynamical map $\Lambda_t$ is non-Markovian according to the BLP condition, when the trace distance $T(\rho,\sigma) = 1/2||\rho-\sigma||_1$ between at least one pair of reduced states is increasing, *i.e.*, there exist times $s, \tau : \tau > s > 0$, so that

$$T(\Lambda_\tau[\rho(0)], \Lambda_\tau[\sigma(0)]) - T(\Lambda_s[\rho(0)], \Lambda_s[\sigma(0)]) = T(\rho(\tau), \sigma(\tau)) - T(\rho(s), \sigma(s)) > 0. \tag{1}$$

The definition was recently generalized to other quantifiers of state distinguishability [7, 8], in particular entropic quantifiers, which is the main topic of this note.

## 2. – Non-Markovianity in terms of different distinguishability quantifiers

There exist many different quantifiers of distinguishability between two quantum states $\rho$ and $\sigma$. The one mostly used in the context of quantum non-Markovianity is the trace distance. As its name indicates, it is a proper mathematical distance, *i.e.*, non-negative and equals zero only for identical states, symmetric in its arguments and satisfies triangle inequality. It also has a measurement interpretation, as it constitutes a minimal probability of error by distinguishing between two equally probable quantum states in a single experiment [9]. Another important property of the trace distance is its contractivity under positive trace preserving operations. With this, its invariance under common unitary transformations and under tensor addition of the common state follows (for the other direction one needs additionally contractivity under partial trace or concavity [10]). As will be shown below, these properties make possible to connect the revivals of trace distance as described by eq. (1) with information back-flow from the environmental degrees of freedom to the reduced system and, based on it, to introduce a corresponding measure of quantum non-Markovianity. However, it is not sensitive to the non-Markovianity coming from the non-unital part of the dynamics since it is a function of the difference between states. What is more, it lacks monotonicity under taking tensor powers of its arguments [10] and according to it the nearest product state for an arbitrary bipartite state is not the tensor product of its marginals [11]. Quantum relative entropy $S(\rho,\sigma) = \mathrm{Tr}(\rho\log(\rho) - \rho\log(\sigma))$ does not suffer from both of the last mentioned problems, consequently it can be seen as a better quantifier of correlations. Additionally, the Hilbert-Schmidt distance $H(\rho, \sigma) = ||\rho - \sigma||_2$ seems to be more intuitive than the trace distance, as it is the Euclidean distance of the coordinate vectors corresponding to the quantum states [12]. As we see, different distinguishability measures (among which we have named only a few) bring different advantages with themselves, and a thorough investigation of properties guaranteeing the existence of an associated definition of quantum non-Markovianity is needed. For a distinguishability quantifier $D(\rho, \sigma)$ such an interpretation of information back-flow exists, when it satisfies, for all times $\tau \geq s \geq 0$:

$$\begin{aligned} &D(\rho(\tau), \sigma(\tau)) - D(\rho(s), \sigma(s)) \leq \\ &f\left(D(\rho_E(s), \sigma_E(s)), D(\rho_{SE}(s), \rho(s) \otimes \rho_E(s)), D(\sigma_{SE}(s), \sigma(s) \otimes \sigma_E(s))\right), \end{aligned} \tag{2}$$

where $f(0,0,0) = 0$ and $D(\rho, \rho) = 0$. In the derivation below, function $f$ will be otherwise positive, this is however not necessary for the interpretation explained below. The positivity of the left-hand side of the above inequality can only be interpreted as a gain of information contained in the reduced degrees of freedom when the distinguishability quantifier $D(\rho, \sigma)$ is contractive under complete positive maps. In particular, subsystems cannot contain more information than the total system. Note, that the invariance under common unitaries is not enough, and, accordingly, the Hilbert-Schmidt distance is not suitable for this purpose.

With eq. (2) from the distinguishability revival in time interval $[s, \tau]$ (positivity of the left-hand side) the existence of information outside the reduced state at previous time $s$ can be concluded. This information can be manifested as a difference in the environmental states corresponding to the two different initial reduced states or as correlations between states $\rho(s)$ or $\sigma(s)$ and their environments. Accordingly, even if the quantum non-Markovianity is a property of the reduced evolution only (see left-hand side of the inequality (2)), one could conclude from its occurrence some properties of the corresponding microscopical realisation.

The derivation for the trace distance is based on its contractivity under positive trace preserving maps and the triangle inequality:

$$\begin{aligned}
&T(\rho(\tau), \sigma(\tau)) - T(\rho(s), \sigma(s)) \leq T(\rho_{SE}(s), \sigma_{SE}(s)) - T(\rho(s), \sigma(s)) && \text{(3a)}\\
&= T(\rho_{SE}(s), \sigma_{SE}(s)) + T(\rho(s) \otimes \rho_E(s), \sigma_{SE}(s)) && \text{(3b)}\\
&\quad - T(\rho(s) \otimes \rho_E(s), \sigma_{SE}(s)) - T(\rho(s) \otimes \rho_E(s), \sigma(s) \otimes \rho_E(s)) &&\\
&\leq T(\rho_{SE}(s), \rho(s) \otimes \rho_E(s)) + T(\sigma_{SE}(s), \sigma(s) \otimes \rho_E(s)) && \text{(3c)}\\
&\leq T(\rho_{SE}(s), \rho(s) \otimes \rho_E(s)) + T(\sigma_{SE}(s), \sigma(s) \otimes \sigma_E(s)) + T(\rho_E(s), \sigma_E(s)), && \text{(3d)}
\end{aligned}$$

where the first was used in eqs. (3a), (3b), (3d) and the latter in eqs. (3c), (3d). Consequently, a function $f$ introduced in eq. (2) is a sum of three non-negative terms, each of which depends only on one of the possibilities of information storage in external degrees of freedom.

It was shown in [8] that, together with boundedness and the contractivity under complete positive maps, fulfilling triangle-like inequalities is sufficient for existence of a definition of quantum non-Markovianity associated with the distinguishability quantifier $D(\rho, \sigma)$ as described above. The triangle-like inequalities read

$$D(\rho, \sigma_1) - D(\rho, \sigma_2) \leq g(D(\sigma_1, \sigma_2)), \qquad D(\sigma_1, \rho) - D(\sigma_2, \rho) \leq g(D(\sigma_1, \sigma_2)), \tag{4}$$

where function $g$ is monotonically non-decreasing, subadditive and $g(0) = 0$. With this, using a similar logic as in the derivation of inequality for trace distance, one obtains:

$$D(\rho(\tau), \sigma(\tau)) - D(\rho(s), \sigma(s)) \leq g \circ g(D(\rho_E(s), \sigma_E(s))) \tag{5a}$$

$$+g(D(\rho_{SE}(s), \rho(s) \otimes \rho_E(s))) + g(D(\sigma_{SE}(s), \sigma(s) \otimes \sigma_E(s))). \tag{5b}$$

Note, that to bound the left-hand side of (2) from above, the corresponding distinguishability quantifier $D(\rho, \sigma)$ has to be finite. This is not the case for quantum relative entropy, and one has to resort to one of its renormalizations. The quantum Jensen-Shannon divergence $J(\rho, \sigma) = 1/2(S(\rho, 1/2(\rho+\sigma)) + S(\sigma, 1/2(\rho+\sigma)))$ can be connected with the Holevo quantity and the telescopic relative entropy [8], and its square root is the only known entropic distinguishability quantifier which is also a proper distance [13,14]. With this derivation of inequality describing information back-flow in terms of square root of the quantum Jensen-Shannon divergence takes the same steps as the one introduced in eq. (3). The telescopic relative entropy $S_\mu(\rho, \sigma) = 1/\log(\mu) S(\rho, \mu\rho + (1-\mu)\sigma)$, $0 < \mu < 1$, is not a distance, however it is indeed bounded and satisfies triangle-like inequalities, and, additionally, inherits contractivity under positive trace preserving maps from the quantum relative entropy [15]. Accordingly, a corresponding definition of quantum non-Markovianity can be introduced and the inequality of the form (5) derived [7].

A next step is to investigate how the choice of different initial states influence the time evolution of different distinguishability quantifiers. Of particular interest is a characterisation of the optimal pair of states which exhibit maximal revivals.

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