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Structure of discontinuities in beta functions in higher-derivative gauge theories

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Summary. — We discuss the structure of perturbative one-loop beta functions in higher-derivative gauge theories, which are quadratic in the field strengths. The action of these theories contains arbitrary integer powers n of the gauge-covariant analogue of the d'Alembert operator. We pay special attention to the discontinuities found for UV-divergences in the cases of $n = 0$ and $n = 1$. They are explained by mathematical properties of the derivation of perturbative vertices and the various usage of finite sum symbols. We also give examples of the derivation of a 3-vertex.

1. – Introduction

The theories with higher derivatives (HD) play an important and very likely crucial role for the formulation of consistent quantum gravitational (QG) theories in $d = 4$ spacetime dimensions. In the gravitational context, they were extensively motivated and studied [1, 2]. Since QG is a very ambitious, but also very difficult topic, one can try to apply the idea of HD to the theories with gauge symmetry, so to the Yang-Mills (YM) theories with non-Abelian gauge symmetries. In this context, HD gauge theories, first were viewed as fully gauge-covariantly regulated versions of two-derivative YM theories [3, 4], such that only one-loop UV-divergences were left present. However, in the modern perspective, HD gauge models are considered more than just toy models for quantum HD gravities. They constitute fully fledged theories of interactions between gluons which entail HD dynamics [4-6] and therefore the behaviour of the theory in the ultraviolet (UV) regime is much more improved. The theories with HD gauge symmetries in $d = 4$ are naturally superrenormalizable [7], unitary [8] and can be easily made also UV-finite [9].

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In HD gauge theories, one can see that the requirement of asymptotic freedom in the UV regime puts very stringent constraints on the form of the theory. Moreover, a hypothetical presence of extra ghosts in HD theories may not harm asymptotically free theories, because in that case the effective masses of such ghost particles under renormalization group (RG) flow are running to infinity in the limit of very high energies [6]. In some models, the perturbative UV-divergences are only at the one-loop level [7, 9].

One can show that the theories with the schematic action written in the form

$$S = \mathrm{tr}(F\Box^n F) \tag{1}$$

with the schematic expansion of the gauge-covariant analogue of the d'Alembertian operator $\Box = \partial^2 + A\partial + (\partial A) + A^2$ are very well behaved in the UV regime, when the integer exponent n is positive. The actions of models in (1) are fully gauge-covariant and they are quadratic in the field strengths F of the non-Abelian gauge theory potentials A. Here, we write them as background quantities and in a schematic index-less notation.

The theories with $n \geqslant 2$ are superrenormalizable and the expressions for the perturbative beta functions of YM gauge coupling are one-loop exact since they cannot receive any higher loop corrections. One also notices that for the case of $n \geqslant 1$, there is no perturbative renormalization of the gluonic wave function. Hence, one can obtain a full information about RG running just reading from one-loop UV-divergences of the theory.

The computation of one-loop beta functions in HD gauge models was first achieved in [4] and also later re-emphasized in [6]. The results reported there were dependent on the exponent n in an interesting and non-trivial way (showing up to quadratic dependence)

$$\beta_g = \frac{g^3 C_2(G)}{32\pi^2} \begin{cases} -\dfrac{22}{3}, & n = 0, \\ \dfrac{38}{3}, & n = 1, \\ 4n^2 + 5n - \dfrac{7}{3}, & n \geqslant 2. \end{cases} . \tag{2}$$

The discontinuity for $n = 0$ and $n = 1$ cases was initially addressed in the appendix A section 1.1.1 of [4]. Here, in this contribution we attempt on extending and completing such explanations and we also give mathematical account of the reasons for such discontinuous jumps in (2) and why the cases of $n = 0$ and $n = 1$ are singled out physically.

In [4], the main results were obtained for $n \geqslant 2$. In the case $n < 2$ such a contribution to UV-divergences cannot be obtained by taking $n = 0$ or $n = 1$ as the limits of the latter. This fact can be easily understood if one looks closely at the 3-leg and 4-leg vertices contributing to the diagram of the vacuum polarization tensor. For $n \geqslant 2$ there is a particular term in these vertices that generates divergences in this one-loop two-point function and it comes from taking as external line one gluon field A from the field strength tensor F, and as the two internal lines one from the F on the left and the other one from the most right gauge-covariant operator $\Box$. Note, that if we took an additional third A-field from any other box operator $\Box$, we would not have a UV-divergence in the two-point correlator, therefore the contribution of this part of the 3-vertex to the divergent piece of the two-point function does not depend on n provided $n \geqslant 1$, but it is absent if $n = 0$. For $n = 1$, there is a similar phenomenon in the perturbative four-vertex.

2. – Discontinuity for $n = 0$ and $n = 1$ cases

Here, we would like to discuss the discontinuity of the final results for UV-divergences in (2) related to these beta functions of the theory at the one-loop level. We first present the reasons for this structure using Feynman diagrams original computation. Later we explain the mathematical intricacies based on some simple example covering all necessary features of the type of computations done here. And next we discuss how the discontinuity arises in other computational approaches to the same problem. Finally, we show an example of the computation of the third variational derivative (3-leg vertex) in higher-derivative gauge theory, paying special attention to the cases of $n = 0$ and $n = 1$.

2·1. *Discontinuity in the computation using Feynman diagrams*. – In the formalism of Feynman diagrams around flat spacetime and gauge-flat space, we see the discontinuity in the final resulting expressions for each of the diagrams. All the steps in the procedure of computing them till the end naively look mathematically continuous and they apparently do not single out special cases of $n = 0$ or $n = 1$. The question is what is the mathematical reason for such discontinuity and how the situation looks on the complex plane of the n parameter, where n is the exponent in the gauge-covariant power of the box operator, this $D_\mu^2 \underset{\text{df}}{=} \Box$, where D_μ is the gauge-covariant derivative. What is the character of this singularity, or what kind of topological obstacle one meets when approaching the special points of $n = 0$ and $n = 1$? The physical reasons and understandings (also in other formalisms of the computation) of this discontinuities are presented to the reader below.

When we write both perturbative vertices and propagators, we see no problem in taking the limits $n \to 0$ or $n \to 1$. In propagators, the higher-derivative regulator factor disappears for $n = 0$, but everything in the derivation of it seems to be done in a continuous way (and the $\Box^n$ is just a spectator for the procedure of finding the propagator). We do not see any obstacle. In vertices we use notation with sums and it is quite clear that they reduce to zero, when there is no summation, so when the upper limit is smaller than the lower one. One also concludes there that the limits are taken smoothly. Then where is the discontinuity coming from?

We can analyze closer the derivation of vertices concentrating on vertices with 3-leg and 4-leg types. When the 3-leg vertex is derived, the only special case is $n = 0$, because then the sums (resulting from the expansion of the kernel $\Box^n$) should not appear. When the 4-leg vertex is analyzed, both cases $n = 0$ and $n = 1$ are special, because in the explicit expression for the 4-leg vertex we have a double sum and the most internal sum should not appear when $n = 1$, and the external one again when $n = 0$. Still, it seems that naively the notation with sums takes care of these special cases and we do not see any problem on this level. One can think here that the vertices are continuous for these limits $n \to 0, 1$ since we use the standard notation with sums, which should cover the cases of presence of terms or their complete absence. However, we should be more careful!

Further algebraic operations are not sensitive to the special cases of $n = 0$ and $n = 1$. We are never particularly in danger in the cases $n = 0$ and $n = 1$ from this point of view. Hence, the reason for discontinuities is even more mysterious. We can describe the situation in more details as follows. Roughly looking in the expressions for perturbative vertices of the theory we have sums in which the number of terms is not fixed, this number grows with n. It is thought that each term in such sums contributes and it is important for final results of UV-divergences. This algebraic proliferation of terms we see especially on the level of perturbative vertices. For 3-leg vertices the number of terms grows linearly with n, while for the 4-leg vertex it grows quadratically (because of the presence of double sums). At the end, we have to somehow "take a trace" of all these terms and get fewer resulting terms, which lead to the final result. These "traces" should have a fixed number of terms (independent of n) in their expanded tensorial expressions. The n-dependence should show up only in front coefficients of these last resulting terms as some polynomial (up to the fourth degree due to the presence of 4-leg vertices). This is a structural change of the intermediate expressions – from the sum with (roughly) n or n^2 elements (without internal n-dependence) to a one term (or a few terms with a fixed number of them), but with overall coefficients depending on n. One could say that this is a result of a tracing of all terms (we show example of this also below). But this process of changing the structure of intermediate expressions of the middle of the computation seems still continuous and we do not see the problem with $n = 0$ or $n = 1$ cases. However, the discontinuity really appears here for the first time and then it is later brought till the end of the computation and inherited in the final discontinuous results for special $n = 0$ and $n = 1$ cases. The situation with $n \geqslant 2$ is analytic and continuous.

A closer analysis shows that we do not really take full trace and not all the terms in the sums are equally important. There is importance of only a few terms, which is a small subset of all terms. These terms are crucial for further computation and for isolating the UV-divergences. Strictly speaking, we extract one given series expansion coefficient from the sum and this process reveals to be discontinuous for the special cases of $n = 0$ or $n = 1$. The series expansion forces us to take, for example, the first term in the sum, which results from the higher-derivative regulator. But what if this sum is not present at all in the first place? Is it consistent to assume that this series coefficient vanishes in such a case? This would be the conclusion forced on us by assumption of the full analyticity in the n variable and from the extended definition and usage of the mathematical symbol of the finite discrete sum $\sum_{i=a}^{b} f_i$ (more explicitly we discuss this issue also below). However, in the expression for the vertex, for example, for the 3-leg vertex, we have a bunch of terms from which we need to extract the given series coefficient. The origin of terms (whether they come from this sum or from another) does not matter for the series expansion process. It is indeed the case, that the series coefficients for $n = 0$ or $n = 1$ are not zero although the corresponding sums vanish. This is the culprit and mathematical/analytical origin of the discontinuity.

2˙2. *Simpler mathematical model*. – One could analyze the similar simplified mathematical model, which shows the essential features of the situation with discontinuities here. The problem lies with the usage of the sum symbol beyond its standard mathematically valid regimes of applicability. Let us analyze the Newtonian binomial expression

$$(x+y)^m$$

and its expansion given for $m \geqslant 0$ by

$$(x+y)^m = \sum_{i=0}^{m} \binom{m}{i} x^i y^{m-i} = \sum_{i=0}^{m} \frac{m!}{(m-i)!i!} x^i y^{m-i}. \tag{3}$$

This formula is still valid for $m = 0$, although then we have just one term, which is unity. The case for m negative (but still integer) is in our biggest interest, because of similarities to the expansion series coefficient of the intermediate expressions constructed both with perturbative propagators and vertices of the higher-derivative model from a previous subsection. One can convince oneself that for Feynman diagram computations, we take effectively like $m = -1$ series coefficients of the expressions, which look very much the same like the Newtonian binomial. The special case of $m = -1$ is paradigmatic here. Assuming that $|\frac{y}{x}| < 1$, we have the exact expression given by the infinite series

$$(x+y)^{-1} = \frac{1}{x}\left(1+\frac{y}{x}\right)^{-1} = \frac{1}{x}\sum_{i=0}^{+\infty}\left(-\frac{y}{x}\right)^{i} = \frac{1}{x}\left(1-\frac{y}{x}+\frac{y^2}{x^2}-\frac{y^3}{x^3}+\dots\right). \tag{4}$$

We are sure that this series is convergent (it is a standard geometric one), when $|\frac{y}{x}| < 1$ and this gives us a correct and exact answer for the case of $m = -1$. What if we want to use our expansion with finite discrete sums as it is in the formula (3) above? First observation is that clearly by finite sums we will not reproduce the infinite series result.

Moreover, here we see that with the finite summation symbol notation as used in (3), the answer is even more ambiguous. Below we use an arbitrary function f_i to denote the general i-th term of the sequence f to make our point. For the case $m = -1$, we have a few options of what the finite sum would look like. First, we know that in the marginal case $m = 0$, the sum is still defined and its result is just one term according to

$$\sum_{i=0}^{0} f_i = f_0\,.$$

What do we have for $m = -1$? When we write sums in the expansion of vertices, we implicitly said and later used that

$$\sum_{i=0}^{-1} f_i = 0\,, \tag{5}$$

so that the sum is not present (vanishes). This would be a hidden assumption for all the sums written in the first part of this section. So we would have assumed that

$$\sum_{i=a}^{b} f_i = 0\,,$$

when $b < a$ and all a, b are integers. This is one possible definition of the extension of the meaning of the symbol $\sum_{i=a}^{b}$ beyond the mathematically original region of validity, where $b \geqslant a$; here with an unspoken assumption about the vanishing of the sum.

On the other hand, one finds the general formula for the addition of sums

$$\sum_{i=a}^{b} f_i + \sum_{i=b+1}^{c} f_i = \sum_{i=a}^{c} f_i \tag{6}$$

and if this one is used for the case of $a = 0$, $b = -1$, and $c > 0$, one gets that

$$\sum_{i=0}^{-1} f_i + \sum_{i=0}^{c} f_i = \sum_{i=0}^{c} f_i\,.$$

Hence in this case, one also finds that $\sum_{i=0}^{-1} f_i = 0$. Obviously, for this one assumes the validity of the addition law (6) no matter what the ordering relation between a, b and c coefficients is. But using the relation in (6) for $a = 0$, $b = -2$ and $c > 0$, one gets that $\sum_{i=0}^{-2} f_i \neq 0$ contrary to our assumption of vanishing of the sum when $b < a$, so when writing the lower and upper limits in relation $b \geqslant a$ is not legitimate. Actually, in this case one finds that $\sum_{i=0}^{-2} f_i = -f_{-1}$, even with the danger that f_{-1} may not be defined.

There could be also plenty of another definitions, one of which, in particular, gives

$$\sum_{i=0}^{-1} f_i = f_{-1}, \text{ etc.} \tag{7}$$

Applying the ambiguous definition $\sum_{i=0}^{-1} f_i = 0$ in (5) to the expression in (3) for the $m = -1$ case, we find an apparently absurd result

$$(x+y)^{-1} = \sum_{i=0}^{m} \binom{m}{i} x^i y^{m-i} = \sum_{i=0}^{m} \frac{m!}{(m-i)!i!} x^i y^{m-i} = \sum_{i=0}^{-1} f_i = 0\,, \tag{8}$$

although it is true that the expression f_i under the sum in (3) was not well defined either in the case $m = -1$, because of the dangerous part $(-1)!$ too. But, if in the sum only $i = 0$ is used, then we could naively simplify the unique term in the summation for $m = -1$ "upper limit" to the form

$$f_i = \frac{m!}{(m-i)!i!} y^{-1} = \frac{(-1)!}{(-1)!0!} y^{-1} \cong \frac{1}{0!} y^{-1} = y^{-1}\,.$$

But the absurd result using (8) is that

$$(x+y)^{-1} = 0\,,$$

while we know that the exact and correct expression is given by the series above in (4). This shows the discontinuity explicitly (here in the shifted case of $m = -1$, away from $m \geqslant 0$ cases, where there are no problems with finite expansion of the Newton binomial).

After all this mathematical discussion, one can understand what the problem was. When we take a series expansion coefficient we also pick up some term from the sums and these kinds of sums are not well defined when they supposed to vanish (for $b < a$). We see that even in Newtonian binomial formula there is a discontinuity in the case of $m = -1$ and also for other negative integers. This is due to overuse of the sum symbol sign. This was done carelessly especially in the situation when the ranges of the summation do not satisfy the proper ordering relation. One should not be misled by the formal appearance of the mathematical symbols like $\sum_{i=0}^{-1} f_i$ and should deal with them with necessary care.

More concretely, implicitly in the argumentation presented at the beginning of this section, we used two different definitions of the sum symbol in the same computation using Feynman diagrams approach. And this was the source of our confusion and of our erroneous hypothesis there that the results should be continuous in the cases of $n = 0$ and $n = 1$. At the beginning, when deriving the vertices we used a definition with the naive hidden assumption exemplified in a statement that $\sum_{i=0}^{-1} f_i = 0$. This lets us conclude that some sums vanish, when they do not appear for lower n cases. But in the middle of the computation we used implicitly a different definition of summing, one similar to the one shown in the definition of the binomial expansion as in (4), which extends also analytically to the cases of m negative there (they correspond to our cases of $n = 0$ and $n = 1$, while $m = 0$ is the same boundary moment as for $n = 2$ in the $n \geqslant 2$ continuous and analytic regime). These two definitions of the usage of the sum symbol do not match and there is a clash between them realizing itself clearly in the discontinuity found in the cases of $n = 0$ and $n = 1$ of the Feynman diagram computation.

By the ambiguous usage of the sum symbol we ran into a problem with its meaning. It is a fact that the first definition was useful for deriving vertices and gave us a nice handle on how to write the expressions for vertices also in the cases of lower n (cf. (5)) by keeping all formulas compact, even in the case of absence of some terms. The second definition was used in the moment, when we did a series expansion (similar to the case of binomial series expansion in (4), which can be analytically continued to the cases of m negative and when the series can be summed analytically, even for the $m = -1$ case).

During the series expansion, we could rely on the analytic properties of our second summation convention and then the cases of $n = 0$ or $n = 1$ are not treated exceptionally and are analytically continuous, provided that the input from vertices is also continuous in the same way, so using the same sum definition. (We know that the input from propagators is always continuous here. The total contribution of diagrams is discontinuous in the second convention for summing only due to the discontinuities in vertices as written using that sum convention.) When being more careful, we should from the start say that the expressions in cases $n = 0$ and $n = 1$ are different and discontinuous from the contributions for other cases of $n \geqslant 2$, which were written using more global definition for the sum symbol (*i.e.*, second definition). Then the discontinuity would be manifest and spotted from the beginning of the computation. This is the mathematical reason in the formalism of writing sums compactly for the discontinuity, which should be traced back to the discontinuities in perturbative vertices of the theory. Of course, in a mathematically consistent computation process we should use only one consistent definition of the sum symbol and is suggested that the best choice is the second definition as in (4).

2˙3. *Other computational approaches*. – Now, we discuss other explanations for the discontinuity, related more to the physical aspects of the higher-derivative models.

First, in the background field method (BFM) formalism, we can perform a similar computation of the UV-divergences leading to the one-loop beta functions of the theory. Then one needs to compute covariantly the second variational derivative of the action with particular attention given to the variation of the regulator kernel $\Box^n$. In a covariant manner, one sees that from the regulator in the $n = 1$ case, one can extract only two gauge fields from the same D_μ^2 operator, while for $n \geqslant 2$ one can extract two gauge fields from two different D_μ^2 operators. Finally, in the case $n = 0$, there is no regulator, so gauge fields cannot be extracted from it. There is a clear discontinuity in the expression for general "vertices" (second variational derivatives on a general gauge background) in these cases $n = 0$, $n = 1$ and the analytic formula in background gauge is expected only when $n \geqslant 2$. This discontinuity is present already on the level of gauge-covariant second variational derivative of the action, which is well known to correspond to the level of perturbative vertices of the theory analyzed around flat-gauge backgrounds.

The same discontinuity was the first time reported in the case of beta function for the cosmological constant in higher-derivative gravitational theories in the models studied by Asorey, Lopez and Shapiro in [10] with $\Box^n$ in the kernel, also computed in the BFM formalism and using the method of Barvinsky-Vilkovisky trace technology for covariant

Hessian-like operators [11]. This has again to do with the structure of the gravitational kernel operator and whether this is one operator in the first power or two or more powers of the same or perhaps of different operators, so there is a difference from where one can extract some fluctuation fields, between the cases of $n = 0$, $n = 1$ and for $n \geqslant 2$.

Finally, the more physical way of understanding the discontinuity has to do with counting of degrees of freedom in an algebraic way in higher-derivative theories. As is well known, the one-loop beta function basically counts the number of such degrees of freedom weighted by their charges. This is the same situation in higher-derivative theories, where we can extract new algebraic degrees of freedom from the powers of the D_μ^2 operator. Similarly, like in the case of BFM formalism, we see that each D_μ^2 operator gives rise to some new sets of degrees of freedom and their number is roughly proportional to n^2 for $n > 1$. For smaller n the extraction of them is more complicated. One could see this by rewriting the theory in an Ostrogradsky way for redundant fields having only two-derivative actions and then counting all degrees of freedom (without taking into account constraints). This is a bit similar to Hamiltonian formulation but again for all algebraic fields (without constraints). One then sees special cases of $n = 1$ and $n = 0$. However, these are not physical degrees of freedom, but only algebraic, participating in virtual loop processes since the beta function is actually sensitive to such counting and to such degrees of freedom. For example, according to the remark in the paper [12] the terms with Weyl curvature (describing genuine spin-2 fluctuations) describe five times more algebraic degrees of freedom than those with scalar curvature (describing genuine spin-0 gravitational fluctuations). The reason for this is that counted in an algebraic way the spin-2 has 5 degrees of freedom, while spin-0 has only one. A similar situation is here analyzed for regulated gauge theory with regulator being higher-derivative operator. The cases of $n = 1$ and $n = 0$ have special expansions in Ostrogradsky way that is not reproduced in the cases of higher n and the counting of algebraic degrees of freedom is different in those cases. Therefore, there must be a clear discontinuity in counting the degrees of freedom in cases $n = 0$ and $n = 1$ and for higher n, namely for $n \geqslant 2$. We think that this is one of the most physical way of understanding this lack of analyticity and the jumps in n-dependence.

The question of what happens in theories with n on the complex plane remains unanswered. Such theories can be defined only by some analytic continuation of the theories we know and like for n being positive integer, so for higher-derivative gauge models. A lot depends here on this continuation. If one continues the analytic function of the polynomial, then one does not have any discontinuity on the complex plane except the mentioned two points $n = 0$ and $n = 1$, which constitute isolated irremovable singularity points. In other points, we define the models by this continuation, so these are smooth continuous functions being analytic extensions of the polynomials in n. In this way, one can define the theory for n non-integer positive. Some examples for such theories are non-local gauge theories, which are generalizations of usual standard gauge theories with higher derivatives as considered here. This is the way the non-local theory in such cases is defined in a first place with the analytic continuation as an essential part of the definition procedure. Other definitions are also possible (through other representations of powers n on D_μ^2 operator for n being non-integer or negative), but they define from the start different models. One could try to remove the singularities in the points $n = 0$ and $n = 1$ and substitute them by the limits from any other directions on the complex planes ($n \to 0$ and $n \to 1$), which are all well defined and unambiguous as limits. However, such "limiting models" of $n \to 0$ and $n \to 1$ cases are very artificial, in a sense also non-local, and moreover they are inconsistent as higher-derivative quantum models of dynamics, since the true models with $n = 0$ (standard two-derivative Yang-Mills gauge theory) and with $n = 1$ (four-derivative gauge theory) are different. Their limiting versions cannot be viewed as physically relevant regularizations of the latter strict $n = 0$ or $n = 1$ models, because we know that physical results, for example, for the beta functions of Yang-Mills theory coincide with the computations in the strict $n = 0$ model and not with the ones performed in the regularized $n \to 0$ "limiting model" obtained as analytic continuation of the general higher-derivative gauge model with $n \geqslant 2$.

The general point here with using various definitions of non-local gauge theories is to use one selected definition of the model consistently for all the theoretical computations (for UV-divergences as well as for studies of exact classical solutions, for instance), which may later show the consistency and usefulness of such a chosen definition. There is no *a priori* universal or unambiguous way to define the theory for n non-integer and non-positive, so none should expect to get results which reproduce some standard results, because such do not exist. Every definition of the model requires some regularization procedure. The results can be only checked for analyticity and consistency of regularization procedure within the given model to the accuracy of higher loop orders.

2˙4. *Example of the computation of the 3-leg vertex.* – Let us consider the following schematic expression X, which for us represents the most essential features of the higher-derivative kernel used in the definition of quadratic gauge theories as in (1):

$$X = (\alpha + \beta\phi)^n$$

and for the three-leg vertex on flat-gauge connection background we need only to find $\frac{\mathrm{d}X}{\mathrm{d}\phi}\Big|_{\phi=0}$ and let us assume that $[\alpha, \beta] \neq 0$. We have the schematic expression for this derivative

$$\frac{\mathrm{d}X}{\mathrm{d}\phi}\bigg|_{\phi=0} = \sum_{i=0}^{n-1} \alpha^i \beta \alpha^{n-1-i} \,. \tag{9}$$

This is an example of the usage of the first definition of the sum. We understand that with this definition, we have in some special cases of summation that

$$\sum_{i=0}^{-1} f_i = 0 \quad \text{and} \quad \sum_{i=0}^{0} f_i = f_0 \,. \tag{10}$$

If the commutator $[\alpha, \beta] = 0$, or for example under the trace, when we find that $\mathrm{tr}\,([\alpha, \beta]) = 0$, then the last expression simplifies to

$$\frac{\mathrm{d}X}{\mathrm{d}\phi}\bigg|_{\phi=0} = n\alpha^{n-1}\beta, \quad \text{so} \quad \mathrm{tr}\left(\frac{\mathrm{d}X}{\mathrm{d}\phi}\bigg|_{\phi=0}\right) = n\alpha^{n-1}\beta \,. \tag{11}$$

In (11), we see only the overall linear n-dependence in front of just one term, while in (9) we have a number of terms (precisely n of them) but each one with unit coefficient, independent of n. This change from (9) to (11) is a structural change of the intermediate expressions that we discussed in sect. **2**˙1 and is the true origin of the discontinuity.

Everything looks continuous in formulas (9) and (11) and the limit $n \to 0$ can be taken smoothly both on the level of sums and also on their traces. However, the problems with continuity arise when one wants to extract some series coefficients from these expressions. The example of this is the formal coefficient in front of the α^{-1} power in (9) and (11).

If the action of the theory is as written in (1), then we can analyze the situation with the perturbative 3-gluon vertex of interactions in more details. Here, in general, the derivatives and background gauge fields satisfy $[\partial, A] = \partial A \neq 0$.

If we want to take the 3-leg vertex, for example, on flat gauge-background, we get

$$\frac{\delta^3}{\delta A^3} S\bigg|_{A=0} \supset 6\left(\frac{\delta F}{\delta A}\frac{\delta}{\delta A}(\Box^n)\frac{\delta F}{\delta A}\right)\bigg|_{A=0} = 6\,\frac{\delta F}{\delta A}\bigg|_{A=0}\frac{\delta}{\delta A}(\Box^n)\bigg|_{A=0}\frac{\delta F}{\delta A}\bigg|_{A=0}$$

$$= 6\,\frac{\delta F}{\delta A}\bigg|_{A=0}\sum_{i=0}^{n-1}\left(\partial^2\right)^i\frac{\delta\Box}{\delta A}\bigg|_{A=0}\left(\partial^2\right)^{n-i-1}\frac{\delta F}{\delta A}\bigg|_{A=0}. \tag{12}$$

The partial contributions to the 3-vertex in (12) are entirely based on the usage of the formula (9) for the first derivative of the $\Box^n$ operator as the kernel between two field strengths F, as this is written in the regulated action in (1). In other contributions to the 3-leg vertex on flat-gauge connection backgrounds (they are $\delta^2 F \Box^n \delta F$ and $\delta F \Box^n \delta^2 F$ respectively), the $\Box^n$ behaves like a spectator and morphs into $\Box^n|_{A=0} = \partial^2$. Here, we use the background with vanishing gauge connection $A = 0$. Of course, this realizes, in particular, the case of invariant condition of being flat-gauge connection with $F = 0$.

In particular, from the action with $n = 0$ (no higher-derivative case), we have

$$\frac{\delta^3}{\delta A^3} S_{n=0}\bigg|_{A=0} \supset 6\,\frac{\delta F}{\delta A}\bigg|_{A=0}\sum_{i=0}^{-1}\left(\partial^2\right)^i\frac{\delta\Box}{\delta A}\bigg|_{A=0}\left(\partial^2\right)^{-i-1}\frac{\delta F}{\delta A}\bigg|_{A=0} = 0\,,$$

because of the first equality in (10), while for $n = 1$ we have an expression for the vertex

$$\frac{\delta^3}{\delta A^3} S_{n=1}\bigg|_{A=0} \supset 6\,\frac{\delta F}{\delta A}\bigg|_{A=0}\sum_{i=0}^{0}\left(\partial^2\right)^i\frac{\delta\Box}{\delta A}\bigg|_{A=0}\left(\partial^2\right)^{-i}\frac{\delta F}{\delta A}\bigg|_{A=0}$$

$$= 6\,\frac{\delta F}{\delta A}\bigg|_{A=0}\left(\partial^2\right)^0\frac{\delta\Box}{\delta A}\bigg|_{A=0}\left(\partial^2\right)^0\frac{\delta F}{\delta A}\bigg|_{A=0} = 6\,\frac{\delta F}{\delta A}\bigg|_{A=0}\frac{\delta\Box}{\delta A}\bigg|_{A=0}\frac{\delta F}{\delta A}\bigg|_{A=0}. \tag{13}$$

For the 3-leg vertex, only the case of $n = 0$ is discontinuous. Instead there for $n \geqslant 1$, as in (13), the expression is normal and continuous. The case of 4-leg vertex makes special cases of $n = 0$ and $n = 1$ since then in its expression there are double sums with the most internal sum being sensitive to the case of $n = 1$ only.

3. – Conclusions

In this contribution, we discussed the origin of the discontinuities in the UV-divergences of HD gauge theories with various exponents n on the gauge-covariant d'Alembertian operator $\Box = D_\mu^2$. They, being related to the beta functions of the gauge coupling at the one-loop level, show jumps at the values of $n = 0$ and $n = 1$. We found the mathematical reasons for them in the discontinuities present in the perturbative vertices of the theory, although naively they could be written using sum symbols which make expressions for them looking seemingly perfectly continuous in the n variable. The same results will apply for the case of the beta functions in quantum gravitational theories with higher derivatives and quadratic in gravitational curvatures (Ricci scalars or Weyl tensors) with the kernel operator being of the same type $\Box^n$, like for the case of $n = 1$ in [13].

* * *

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