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Regularization of Newtonian-limit singularities in higher-derivative gravity models

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Summary. — In this talk we review some recent results concerning the occurrence of regular solutions in local and non-local higher-derivative gravity models. We show that, even though fourth-order gravity still has curvature singularities, any local model with at least six derivatives in the spin-2 and spin-0 sectors has a regular Newtonian limit, without curvature singularities, when coupled to a point-like source. Also, we discuss the general conditions for the regularity of the higher-order curvature invariants, in both local and non-local models, in the linearised limit.

1. – Introduction: Higher-derivative gravity models

The main motivation for the introduction of higher derivatives in gravity theories comes from the fact that general relativity (GR) is not perturbatively renormalisable, and higher-derivative terms occur already in semi-classical gravity [1]. Indeed, in the renormalisation of a quantum field theory in curved space-time, higher-derivative terms with four derivatives of the metric appear as divergences associated with vacuum diagrams. Besides that, fourth-derivative gravity is a renormalisable theory, as shown by Stelle in 1977 [2]; while higher-derivative gravity (HDG) with six and more derivatives can be super-renormalisable [3]. As it has already been discussed in this conference, higher derivatives bring the problem of ghosts, and a definitive answer on how to deal with the ghosts, or how to live with them, or how to make them harmless, is still needed. There are many proposals in these directions (some of them were already discussed in the conference), for example, Lee-Wick gravity [4, 5] (based on HDG with complex poles), the "fakeon" quantisation prescription by Anselmi and Piva [6, 7], the possibility of having unstable ghosts [8], and the idea of avoiding ghosts from the beginning by means of non-locality [9-14]. We are not going to elaborate on the ghost problem here —instead, we are going to consider only the issue of the Newtonian singularities in these models.

The general gravitational action we are going to consider can be written in the form

$$S = -\frac{1}{\varkappa^2}\int \mathrm{d}^4x\sqrt{-g}\left[2R + C_{\mu\nu\alpha\beta}F_2(\Box)C^{\mu\nu\alpha\beta} - \frac{1}{3}RF_0(\Box)R\right], \tag{1}$$

where $C_{\mu\nu\alpha\beta}$ is the Weyl tensor and $\varkappa^2 = 32\pi G$. The first term corresponds to the Einstein-Hilbert action, while the other two terms contain four or more derivatives, or even an infinite number of them if it is to describe a non-local theory. As we are concerned only with singularities —actually, one of the simplest singularities in Physics, that is the Newtonian singularity (or the singularity associated with a Dirac delta source)— the action above is the most general for this purpose. One could add, for example, a term quadratic in the Riemann tensor, but then it is possible to recast the action in the form (1) plus a term of order (curvature)^3 that is not relevant at the Newtonian limit, just like other terms of higher power in curvatures.

Let us now recall some classical results regarding theories of the type (1). If the functions $F_{0,2}$ are constants, we have the fourth-derivative gravity, whose modified Newtonian potential,

$$\varphi(r) = -\frac{GM}{r}\left(1 - \frac{4}{3}e^{-m_{(2)}r} + \frac{1}{3}e^{-m_{(0)}r}\right), \tag{2}$$

is finite [2]. When $r \to 0$ the combination of exponential functions converges to -1, cancelling the singularity. In fact, the potential can be expanded as a Taylor series around $r = 0$, with a constant term and higher powers on r. However, the finiteness of the potential is not enough to prevent the occurrence of singularities in curvature invariants. For example, still in the linear approximation, you can evaluate the Kretschmann scalar associated with a Dirac delta source,

$$R^2_{\mu\nu\alpha\beta} \underset{r\to 0}{\sim} \frac{G^2M^2}{r^2}(m^4_{(0)} + m^2_{(0)}m^2_{(2)} + 7m^4_{(2)}), \tag{3}$$

that still diverges. (These calculations are all in the linear regime. Considerations in the non-linear regime are much more involved and have been done, *e.g.*, by Stelle and collaborators [15,16], and the final result is that the static spherically symmetric solutions that couple to a delta source still have curvature singularities.)

Another result that is worth mentioning is that in the incomplete polynomial gravity (that is, the action (1) with $F_2 = 0$ but F_0 being a polynomial) the potential is not finite [17].

Our motivation here is to review some more recent results regarding the occurrence of singularities in the Newtonian limit in theories of the type (1). It is very complicated to carry out the general analysis in the full non-linear theory. So, we are going to focus on the simplest singularities, that are related to the Newtonian limit and the field generated by a point-like source.

2. – Newtonian limit

In the Newtonian limit we can take the metric as a small perturbation around the Minkowski space-time, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and expand the action up to second order on

$h_{\mu\nu}$. The resulting equations of motion, with a source $T^{\mu\nu}$, are

$$H^{\mu\nu,\alpha\beta}\, h_{\alpha\beta} = -\frac{\varkappa^2}{2} T^{\mu\nu}, \tag{4}$$

with

$$\begin{aligned} H^{\mu\nu,\alpha\beta} &= f_2(\Box)\left[\delta^{\mu\nu,\alpha\beta}\Box - \left(\eta^{\alpha(\mu}\partial^{\nu)}\partial^{\beta} + \eta^{\beta(\mu}\partial^{\nu)}\partial^{\alpha}\right)\right] \\ &\quad - \frac{1}{3}\left[f_2(\Box) + 2f_0(\Box)\right]\left[\eta^{\mu\nu}\eta^{\alpha\beta}\Box - \left(\eta^{\mu\nu}\partial^{\alpha}\partial^{\beta} + \eta^{\alpha\beta}\partial^{\mu}\partial^{\nu}\right)\right] \\ &\quad + \frac{2}{3}\left[f_2(\Box) - f_0(\Box)\right]\frac{\partial^{\mu}\partial^{\nu}\partial^{\alpha}\partial^{\beta}}{\Box}, \end{aligned} \tag{5}$$

The functions f_s $(s = 0, 2)$ are related to the form factors F_s in the action,

$$f_s(\Box) = 1 + F_s(\Box)\Box. \tag{6}$$

If these functions are polynomials we have a local HDG, if they are non-polynomial we have a non-local model. The form factors usually considered in non-local HDG have the structure [9-12]

$$F_s(\Box) = \frac{e^{H_s(\Box)} - 1}{\Box}, \tag{7}$$

where H_s is an entire function. Therefore, f_s is the exponential of an entire function and does not have zeros in the complex plane.

Notice that the roots of the equations $f_s(k^2) = 0$ define the massive poles of the propagator associated with the wave operator (5), namely,

$$G_{\mu\nu\alpha\beta}(k) = \frac{P^{(2)}_{\mu\nu\alpha\beta}}{k^2 f_2(k^2)} - \frac{P^{(0-s)}_{\mu\nu\alpha\beta}}{2k^2 f_0(k^2)}, \tag{8}$$

where $P^{(2)}$ and $P^{(0-s)}$ are the spin-2 and spin-0 projectors [18]. Therefore, if the form factor (7) is used, there are no other poles in the propagator besides the usual massless graviton. On the other hand, in a local (polynomial) theory there might be many massive degrees of freedom, depending on the order of the polynomial.

Since we are interested in evaluating the modifications owed to higher derivatives in the Newtonian limit, we need to evaluate the field generated by a point-like mass in rest([1]). Thus, we consider a Dirac delta as source with mass M and assume the metric to be static and spherically symmetric,

$$\mathrm{d}s^2 = (1 + 2\varphi)\mathrm{d}t^2 - (1 - 2\psi)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{9}$$

Hence, the solution to the potentials φ and ψ come from solving (4). This is equivalent to solve [21]

$$f_s(-\Delta)\Delta\chi_s = \kappa_s\rho\,, \qquad \text{with} \qquad \kappa_s = 2(3s-2)\pi G \qquad (s = 0, 2) \tag{10}$$

([1]) For considerations regarding a collapsing null shell, see refs. [19-21].

for $\chi_{0,2}$ and then substitute into the definition

$$\varphi = \frac{1}{3}(2\chi_2 + \chi_0)\,, \qquad \psi = \frac{1}{3}(\chi_2 - \chi_0). \tag{11}$$

It is useful to work with the auxiliary potentials χ_s instead of the original potentials φ and ψ of the metric because each of them only depend on the spin-s part of the propagator.

So, once the equations for $\chi_{0,2}$ are solved, you can obtain the usual potentials in the original presentation. With no loss of generality, in what follows we concentrate the discussion on the potentials χ_s. Using the Fourier transform method we can write the solution to the potential,

$$\chi(r) = -\frac{\kappa M}{2\pi^2 r}\int_0^\infty \frac{\mathrm{d}k}{k}\frac{\sin(kr)}{f(k^2)}, \tag{12}$$

where we dropped the s-label for the sake of simplicity.

Some examples are in order. i) In the fourth-derivative gravity we have the potential [2]

$$\chi(r) = -\frac{\kappa M}{4\pi r}(1 - e^{-mr}), \tag{13}$$

that is the usual Newton plus a Yukawa potential correction with a certain mass m.

ii) In a polynomial-derivative gravity with N simple poles in the propagator there appear more Yukawa-like potentials [22, 23] (see also [17]),

$$\chi(r) = -\frac{\kappa M}{4\pi r}\left(1 - \sum_{i=1}^{N} C_i\, e^{-m_i r}\right), \tag{14}$$

and the constants C_i depend on the masses m_i of the particles in the spectrum (the zeros of (6)),

$$C_i = \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{m_j^2}{m_j^2 - m_i^2}. \tag{15}$$

iii) In the simplest non-local gravity (with $f(k^2) = e^{k^2/m^2}$) the potential is related to the error function [13, 14],

$$\chi(r) = -\frac{\kappa M}{4\pi r}\,\mathrm{erf}\left(\frac{mr}{2}\right) \tag{16}$$

It is almost easy to see that these three potentials are finite at $r = 0$. Maybe the only tricky one is the polynomial case, because here we have all the coefficients C_i and, when $r = 0$, you have to prove that the sum of them all is equal to 1. The proof of this general identity can be found in the work [23], and indeed the singularity is cancelled.

The formulas above for the potential of a general polynomial-derivative gravity are valid also if the propagator has complex poles [23], which is one of the basis of Lee-Wick gravity [4, 5]. It is interesting that these complex poles generate oscillating contributions to the potential, which are suppressed by Yukawa factors [23]. Some applications of these oscillating modes can be found, *e.g.*, in [24-26].

Using the identity $\sum C_i = 1$, it was shown in [23] that the metric potentials φ and ψ are finite for any complete polynomial HDG, that is, a model with higher derivatives in both spin-0 and spin-2 sectors. However, the actual evaluation of the potential for general models, which could include degenerate poles, was done later, in [21].

3. – Regularity of curvature invariants

The finiteness of the metric does not imply that the curvature scalars are going to be regular. For instance, for metric potentials expanded like

$$\begin{aligned}\varphi(r) &= \varphi_0 + \varphi_1 r + \varphi_2 r^2 + \varphi_3 r^3 + O(r^4),\\ \psi(r) &= \psi_0 + \psi_1 r + \psi_2 r^2 + \psi_3 r^3 + O(r^4),\end{aligned} \tag{17}$$

the Kretschmann scalar reads (c_0 is a constant)

$$R^2_{\mu\nu\alpha\beta} = \frac{8(\varphi_1^2 + 3\psi_1^2)}{r^2} + \frac{32(\varphi_1\varphi_2 + 4\psi_1\psi_2)}{r} + c_0 + O(r). \tag{18}$$

It is cursory to realize that there are divergences depending only, basically, on the terms linear on r in the series expansion of the potentials (17) [20]. Thus, a condition to have the regularity of curvature-squared scalars is that $\chi'_{0,2}/r$ goes to a constant when $r \to 0$. Or, if the potential is analytic, as assumed here, that the coefficient of the terms linear on r is null. It is possible to show that the same condition holds for C^2, $R^2_{\mu\nu}$, R^2 and so on (see, *e.g.*, [21]). So, we need to check the terms of order r to see if the singularity appears in curvature invariants or not.

In the work [21] the potential for a general polynomial-derivative gravity was obtained (including the cases of complex poles and/or degenerate poles; the only assumption is the absence of tachyons),

$$\chi(r) = -\frac{\kappa M}{4\pi r} + \frac{\kappa M}{4\pi^{3/2}} \sum_{i=1}^{N} \sum_{j=1}^{\alpha_i} A_{i,j} \left(\frac{r}{2m_i}\right)^{j-\frac{3}{2}} K_{j-\frac{3}{2}}(m_i r)\,. \tag{19}$$

$A_{i,j}$ are constants depending on the masses m_i, while α_i is the multiplicity of each of the N roots m_i. The result of expanding this potential around $r = 0$ is

$$\chi(r) = \frac{\kappa M}{4\pi r}\left[-1 + \sum_{i=1}^{N} A_{i,1}\right] + \chi^{\{0\}} + \chi^{\{1\}} r + \chi^{\{2\}} r^2 + O(r^3), \tag{20}$$

with

$$\chi^{\{1\}} = \frac{\kappa M}{8\pi}(S_1 - S_2), \qquad S_1 \equiv \sum_{i=1}^{N} A_{i,1} m_i^2, \qquad S_2 \equiv \sum_{i=1}^{N} A_{i,2}. \tag{21}$$

One can show that if f is a non-constant polynomial, *i.e.*, if you have a theory with at least fourth derivatives, then $\sum_i A_{i,1} = 1$ and the potential χ is regular. Moreover, if

f is of order larger than 1 (*i.e.*, if you have at least six derivatives in the action), then $S_1 = S_2$ and the linear term vanishes(2). So, if you have a complete theory with at least four derivatives, the modified Newtonian potential is finite; and if you have at least six derivatives the Newtonian limit is regular, without curvature singularities. This holds independently of the exact number of poles in the propagator, nor on whether they are degenerate or complex/real.

We are going to call a potential without these $1/r$ diverging terms as *0-regular* (that is, regular in the standard sense), and the potential that does not have the term linear on r, *1-regular* [27]. So, we can say that HDG with fourth-derivatives in both spin-2 and spin-0 sectors have regular potentials, and those with at least sixth derivatives have 1-regular potentials. The 1-regularity of both potentials χ_0 and χ_2 means that there are no curvature singularities.

4. – Regularity in non-local models

We showed that all complete local models with more than four derivatives have remarkable regularity properties. And now we would like to extend it to non-local models; since they are defined by an infinite series on the d'Alembertian, they might be regarded like a polynomial with infinite order. Of course, this is not an accurate way of thinking, but it can be inspiring in some sense.

Some non-local ghost-free models are known to have a regular Newtonian limit, without curvature singularities. There are many papers containing calculations with specific models (see, *e.g.*, [20, 28-30]). So, our idea was to extend the result we have for general local models to the non-local ones. To this end, we used the effective source formalism. The basic idea is to rewrite eq. (10) for the potentials χ_s in an equivalent form, namely, as the usual Poisson equation with a modified source,

$$\Delta\chi = \kappa\,\rho_{\text{eff}}, \qquad \text{with} \qquad \rho_{\text{eff}}(r) = \frac{M}{2\pi^2}\int_0^\infty \mathrm{d}k\,\frac{k\sin(kr)}{r f(k^2)}. \tag{22}$$

Therefore, instead of a Dirac delta source here we have a smeared distribution. In [31] we showed that the polynomial theories with more than four derivatives have a regular modified source ρ_{eff}, and this makes the potential to be 1-regular (*i.e.*, without the term linear on r). Moreover, in some classes of non-local theories, you can view the corresponding effective source as the limit of a sequence of sources associated with local theories [31]. This holds, for example, in the "ghost-free gravity of type N", defined by $f(k^2) = \exp{(k^2/m^2)^N}$ [28]. In short, since each order of truncation of the series of $f(k^2)$ is a polynomial function, it is related to a regular smeared source. It is possible to show that this sequence of sources converges uniformly, so that the limit source is also regular. Of course, it is not true that this holds for *all* non-local HDG models. For example, there are theories defined by form factors that tend to polynomials in the ultraviolet (UV) for which the convergence may not be uniform (see, *e.g.*, [31] for one such example).

One important conclusion at this point is that the regularity of curvature scalars is not consequence of non-locality or ghost-free condition. There are local theories and

(2) The proof of these identities follows from using partial fraction decomposition and can be found in the appendix of [21].

theories with ghosts that have regular curvature invariants. Actually, it is consequence of the propagator having an improved behaviour in the UV.

5. – Higher-order regularity

More recently we studied the higher-order regularity of the metric. This is motivated by the observation that the absence of the linear term in the series expansion of the potentials is enough to regularise the curvature invariants without covariant derivatives of curvatures —but it is not enough to regularise scalars containing derivatives of the curvatures. For example, for the metric (9) we have

$$\Box^n R = 2\Delta^{n+1}\chi_0 = 2\chi_0^{(2n+2)} + \frac{4(n+1)}{r}\chi_0^{(2n+1)}. \tag{23}$$

As before, χ_0 is the potential related to the spin-0 part of the propagator. Owed to this relation, it is easy to see the existence of a singularity which comes from the $(2n+1)$-th derivative of χ_0. So, if $\chi_0^{(2n+1)}(0)$ is non-zero, the singularity appears again, in the scalar $\Box^n R$.

The result we found in [27] is that if the action is complete with $2N+6$ derivatives of the metric, then all curvature-derivative invariants with at most $2N$ covariant derivatives are regular, and there are singular scalars with $2N+2$ covariant derivatives. For example, if $N = 0$ (sixth-derivative gravity), the first non-zero odd term of the series expansion of the potential is $O(r^3)$ then $\Box R$ diverges. If $N = 1$ (eight-derivative gravity), the potential does not have linear term, nor r^3-term; there is only $O(r^5)$ and so on, so $\Box^2 R$ diverges, etc.

In the case of polynomial theories with only simple poles in the propagator this may be proved directly by using partial fraction decomposition and the expression of the potential (14) [27]. Then, one can see that the coefficients of the odd-power terms are null, up to a certain order. In this spirit, the previous relations that we showed in sect. **3** appear as particular cases. The result is that if the action has $2N+2$ derivatives in the spin-s sector, the potential χ_s is $(N-1)$-regular but it is not N-regular. So, there is a certain minimal number of derivatives that you need to apply in some curvature scalars in order to find some singular invariants.

The general theorem reads: if the potentials $\chi_{0,2}$ are $(p+1)$-regular, then all the linearised curvature-derivative invariants with at most $2p$-derivatives are non-singular. The details can be found in [27].

This result can be extended to higher-derivative non-local theories (also with some non-analytic form factors) by using the effective source formalism, which again proves to be very convenient. But in [27] instead of taking limits and verifying the uniform convergence of the series, we studied the conditions under which the damping caused by form factors in the UV allowed us to interchange some limits. The conclusion is that (see [27] for more details): if the function $f(k^2)$ asymptotically grows at least as fast as k^{4N+2} for a certain N, then the effective source is at least $2N$ times differentiable and it is N-regular. Then, you can extend the regularity of the source to the potential.

It follows, for example, that the potential in non-local theories which tend to polynomials in the UV are not infinitely regular; the explicit example showed in the talk can be found in [27]. See also [32] for the example of fourth-derivative gravity with logarithmic quantum corrections.

6. – Conclusions

In the theories with *more* than four derivatives in both spin-2 and spin-0 sectors we have not only a finite modified Newtonian potential, but also a regular non-relativistic limit, that is, without curvature singularities. Hence, in some sense, quantum and classical singularities are softened when going from 2 to 4, and then to 6 and more derivatives, as we go from GR (non-renormalisable, singularity in the potential and curvature) to fourth-derivative gravity (renormalisable, finite potential but curvature singularities) and to sixth and higher-derivative gravity (super-renormalisable, finite potential, regular curvature).

Concerning the scalars containing covariant derivatives of the curvature tensors, higher-order regularity can be achieved depending on the number of derivatives in the action. It is important to have in mind that the regularity of the curvature invariants is not a consequence of non-locality or a ghost-free condition, but of the behaviour of the propagator in the UV.

"Infinite-order regularity" can be achieved in some types of non-local theories (but not in all of them). In a similar way, the effective regularisation of the Dirac delta source can be achieved in local theories (with six or more derivatives). However, to completely regularise the delta source (that is, to obtain a source whose derivatives are all bounded) one needs a form factor that grows faster than any polynomial, *i.e.*, non-locality is necessary.

All the considerations above are valid in the linear regime, and we believe that these findings motivate the investigation of higher-derivative theories in the full non-linear regime. In this vein, to our best knowledge, the only work on models with six and more derivatives is [33]. Some recent applications of higher-order regularity of regular black holes, and some generalizations of results mentioned in sect. **5** to the non-linear regime can be found in [34].

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REFERENCES

[1] Utiyama R. and DeWitt B. S., *J. Math. Phys.*, **3** (1962) 608.
[2] Stelle K. S., *Phys. Rev. D*, **16** (1977) 953.
[3] Asorey M., López J. L. and Shapiro I. L., *Int. J. Mod. Phys. A*, **12** (1997) 5711, arXiv:hep-th/9610006.
[4] Modesto L. and Shapiro I. L., *Phys. Lett. B*, **755** (2016) 279, arXiv:1512.07600.
[5] Modesto L., *Nucl. Phys. B*, **909** (2016) 584, arXiv:1602.02421.
[6] Anselmi D. and Piva M., *JHEP*, **06** (2017) 066, arXiv:1703.04584; *Phys. Rev. D*, **96** (2017), arXiv:1703.05563.
[7] Anselmi D., *JHEP*, **06** (2017) 086, arXiv:1704.07728.
[8] Donoghue J. F. and Menezes G., *Phys. Rev. D*, **100** (2019) 105006, arXiv:1908.02416.
[9] Krasnikov N. V., *Theor. Math. Phys.*, **73** (1987) 1184; *Teor. Mat. Fiz.*, **73** (1987) 235.
[10] Kuz'min Yu. V., *Sov. J. Nucl. Phys.*, **50** (1989) 1011; *Yad. Fiz.*, **50** (1989) 1630.

[11] Tomboulis E. T., *Superrenormalizable gauge and gravitational theories*, arXiv:hep-th/9702146.
[12] Modesto L., *Phys. Rev. D*, **86** (2012) 044005, arXiv:1107.2403.
[13] Tseytlin A. A., *Phys. Lett. B*, **363** (1995) 223, arXiv:hep-th/9509050.
[14] Siegel W., *Stringy gravity at short distances*, arXiv:hep-th/0309093; Biswas T., Mazumdar A. and Siegel W., *J. Cosmol. Astropart. Phys.*, **03** (2006) 009, arXiv:hep-th/0508194; Biswas T., Gerwick E., Koivisto T. and Mazumdar A., *Phys. Rev. Lett.*, **108** (2012) 031101, arXiv:1110.5249.
[15] Stelle K. S., *Gen. Relativ. Gravit.*, **9** (1978) 353.
[16] Lü H., Perkins A., Pope C. N. and Stelle K. S., *Phys. Rev. Lett.*, **114** (2015) 171601, arXiv:1502.01028; *Phys. Rev. D*, **92** (2015) 124019, arXiv:1508.00010.
[17] Quandt I. and Schmidt H. J., *Astron. Nachr.*, **312** (1991) 97, arXiv:gr-qc/0109005.
[18] Barnes K. J., *J. Math. Phys.*, **6** (1965) 788; Rivers R. J., *Nuovo Cimento*, **34** (1964) 387.
[19] Frolov V. P., Zelnikov A. and de Paula Netto T., *JHEP*, **06** (2015) 107, arXiv:1504.00412.
[20] Frolov V. P., *Phys. Rev. Lett.*, **115** (2015) 051102, arXiv:1505.00492.
[21] Giacchini B. L. and de Paula Netto T., *Eur. Phys. J. C*, **79** (2019) 217, arXiv:1806.05664.
[22] Modesto L., de Paula Netto T. and Shapiro I. L., *JHEP*, **04** (2015) 098, arXiv:1412.0740.
[23] Giacchini B. L., *Phys. Lett. B*, **766** (2017) 306, arXiv:1609.05432.
[24] Accioly A., Giacchini B. L. and Shapiro I. L., *Phys. Rev. D*, **96** (2017) 104004, arXiv:1610.05260.
[25] Boos J., *Int. J. Mod. Phys. D*, **27** (2018) 1847022, arXiv:1804.00225.
[26] Buoninfante L. and Giacchini B. L., *Phys. Rev. D*, **102** (2020) 024020, arXiv:2005.05355.
[27] Burzillà N., Giacchini B. L., de Paula Netto T. and Modesto L., *Eur. Phys. J. C*, **81** (2021) 462, arXiv:2012.11829.
[28] Frolov V. P. and Zelnikov A., *Phys. Rev. D*, **93** (2016) 064048, arXiv:1509.03336.
[29] Edholm J., Koshelev A. S. and Mazumdar A., *Phys. Rev. D*, **94** (2016) 104033, arXiv:1604.01989.
[30] Buoninfante L., Koshelev A. S., Lambiase G. and Mazumdar A., *J. Cosmol. Astropart. Phys.*, **09** (2018) 034, arXiv:1802.00399; Buoninfante L., Koshelev A. S., Lambiase G., Marto J. and Mazumdar A., *J. Cosmol. Astropart. Phys.*, **06** (2018) 014, arXiv:1804.08195.
[31] Giacchini B. L. and de Paula Netto T., *J. Cosmol. Astropart. Phys.*, **1907** (2019) 013.
[32] Burzillà N., Giacchini B. L., de Paula Netto T. and Modesto L., *Phys. Rev. D*, **103** (2021) 064080, arXiv:2012.06254.
[33] Holdom B., *Phys. Rev. D*, **66** (2002) 084010, arXiv:hep-th/0206219.
[34] Giacchini B. L., de Paula Netto T. and Modesto L., *Phys. Rev. D*, **104** (2021) 084072, arXiv:2105.00300.