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Testable upper bound on $\rho = \mathrm{Re}\, f / \mathrm{Im}\, f$

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Summary. — At LHC energies we reach a point where $\ln(s/m^2) > 10$. In this paper we give upper bounds on $\rho = \mathrm{Re}\, f / \mathrm{Im}\, f$ that have no unknown constants and are thus experimentally testable.

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1. – Introduction

For almost half a century, particle physicists have used the two words "High Energy" when writing about particle scattering. This happened as the available collision energy increased by orders of magnitude. But with the birth of the LHC we have reached a new level which can be characterized by new words. Namely the logarithm of the energy in now large, $\ln(s/m^2) > 10$, with the scale m being the mass of the proton.

This opens a significant opportunity for us. Rigorous inequalities derived from axiomatic quantum field theory are now testable by experiment.

In this talk we will focus on one of these inequalities proved by T. Kinoshita and myself in ref. [1]. In that paper we derived an upper bound on $\rho = \mathrm{Re}\, f / \mathrm{Im}\, f$, where f is the forward scattering amplitude. The bound has no unknown constants. It can thus be tested for $\ln(s/m^2) > 10$, in the case of proton-proton scattering.

In sect. **2**, we review the general results of ref. [1], and in sect. **3** we apply these methods to obtain a bound on ρ.

2. – General results

In this section we summarize the general results of ref. [1].

For the forward scattering amplitude of two scalar neutral particles, $f(E)$, with E being the energy in the rest frame of one of the particles, *i.e.*

$$s \equiv 2m^2 + 2mE, \tag{2.1}$$

where s is the square of total C.M. energy, we have

$$f(E) - f(0) = \frac{2E^2}{\pi}\int_{E_0}^{\infty} \mathrm{d}E' \frac{\mathrm{Im}\, f(E')}{E'(E'^2 - = E^2)}\,. \tag{2.2}$$

Here we have assumed that the two particles have no bound state. This so-called "dispersion relation" is a rigorous result of axiomatic quantum field theory.

We now define a function $g(E)$, for $\mathrm{Im}\, E \geq 0$, as

$$g(E) = \int_0^E \mathrm{d}E' \frac{f(E') - f(0)}{(E')^2}\,. \tag{2.3}$$

In ref. [1], the following theorem was proved:

Theorem. *$g(E)$ is univalent for* $\mathrm{Im}\, E \geq 0$*.*

This means that g maps the upper half plane, $\mathrm{Im}\, E \geq 0$, in a one to one manner onto a domain in the *g-plane*.

This property of univalence imposes strong conditions on $g(E)$. From eq. (82) of ref. [1], we get for $E_2 \gg E_1$,

$$\frac{\mathrm{Re}\, g(E_2)}{\mathrm{Im}\, g(E_2)} \leq \frac{\pi}{2}\left[\frac{1}{\left[\ln \frac{E_2}{E_1} - 4\ln 2\right]}\right]. \tag{2.4}$$

3. – Proton-proton scattering

We start with

$$f(E) \equiv \frac{1}{2}\left[f(\bar{p}p) + f(pp)\right]. \tag{3.1}$$

This leads to an even function of E.

Then with $m < E_0 < 2m$, we write

$$f(E) - f(0) = \frac{2E^2}{\pi}\int_{E_0}^{\infty} \mathrm{d}E' \frac{\mathrm{Im}\, f(E')}{E'(E'^2 - E^=2)} + R(E). \tag{3.2}$$

For large energy R is negligible,

$$\left|\frac{R(E)}{\mathrm{Im}\, f(E)}\right| = O\left(\frac{1}{E}\right), \quad E \gg E_0. \tag{3.3}$$

As in ref. [1], we set

$$g(E) \equiv \int_0^E \mathrm{d}E' \frac{[f(E') - f(0)]}{(E')^2}\,; \quad \mathrm{Im}\, E \geq 0. \tag{3.4}$$

Now for $|E| \gg m$, and $m < E_1 < 10m$, we define $\tilde{g}(E)$ as

$$\tilde{g}(E) \equiv \int_{E_1}^{E} \mathrm{d}E' \frac{f(E')}{(E')^2} \,. \tag{3.5}$$

Then it is easy to show that, $(\tilde{g} - g) \to 0$ as $|E| \to \infty$, indeed

$$\tilde{g}(E) - g(E) = O\left(\frac{1}{|E|}\right), \tag{3.6}$$

for large $|E| \gg 10m$.

For $E \gg E_1$, we have, as in ref. [1] and eq. (2.4) above:

$$\frac{\mathrm{Re}\, \tilde{g}(E)}{\mathrm{Im}\, \tilde{g}(E)} \le \frac{\pi}{2} \left\{ \frac{1}{\left[\ln \frac{E}{E_1} - 4\ln 2\right]} \right\}. \tag{3.7}$$

From eq. (3.5), we have

$$\frac{\mathrm{Re}\, \tilde{g}(E)}{\mathrm{Im}\, \tilde{g}(E)} = \frac{\int_{E_1}^{E} \mathrm{Re}\, f(E')\mathrm{d}E'/E'^2}{\int_{E_1}^{E} \mathrm{Im}\, f(E')\mathrm{d}E'/E'^2} \,. \tag{3.8}$$

However, we can write

$$\begin{aligned} \mathrm{Re}\, f &= \rho\, \mathrm{Im}\, f, \\ \mathrm{Im}\, f &= cE\sigma(E), \end{aligned} \tag{3.9}$$

where σ is the total cross section, and c a constant. This leads us to

$$\frac{\mathrm{Re}\, \tilde{g}}{\mathrm{Im}\, \tilde{g}} = \frac{\int_{E_1}^{E} \rho(E')\sigma(E')\mathrm{d}E'/E'}{\int_{E_1}^{E} \frac{\sigma(E')}{E'}\mathrm{d}E'} \,. \tag{3.10}$$

The constant, c cancels out.

Substituting eq. (3.10) in (3.7) we obtain,

$$\frac{\int_{E_1}^{E} \rho(E')\sigma(E')\mathrm{d}E'/E'}{\int_{E_1}^{E} \frac{\sigma(E')}{E'}\mathrm{d}E'} \le \frac{\pi}{2} \left\{ \frac{1}{\left[\ln \frac{E}{E_1} - 4\ln^2\right]} \right\}. \tag{3.11}$$

Here again $m < E_1 < 10m$. This last bound has no unknown constants and ρ and σ are experimentally measurable. The violation of this bound by the data leads to really new physics.

Finally if we take the case where the Froissart bound is reached, *i.e.* $\sigma \cong b\big(\ln \frac{E}{m}\big)^2$, and set the scale $m = 1$, we get

$$\frac{\mathrm{Re}\, \tilde{g}(E)}{\mathrm{Im}\, \tilde{g}(E)} \cong \frac{3}{(\ln E)^3} \int_{E_1}^{E} \frac{\mathrm{d}E'}{E'} (\ln E')^2 \rho(E'). \tag{3.12}$$

Hence, from eqs. (3.10) and (3.11) we obtain

$$\int_{E_1}^{E} \frac{\mathrm{d}E'}{E'} \rho(E')(\ln E')^2 \leq \frac{\pi}{6} \left\{ \frac{(\ln E)^3}{\left[\ln\left(\frac{E}{E_1}\right) - 4\ln 2\right]} \right\}. \tag{3.13}$$

A quick comparison with available data on ρ shows that this bound is close. However, this author prefers to wait for more data on ρ coming from different experimental groups.

We hope this will happen before the next La Thuile Conference.

REFERENCES

[1] N. N. Khuri and T. Kinoshita, *Phys. Rev. B*, **140** (1965) 706.