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Statistical approach to Sommerfelds fine-structure constant(*)

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Summary. — This study considers the entropy at the interaction of electrons via photons. Due to full reversibility, a conservation law for the interaction entropy is assumed. The interaction entropy for classical particles, as well as particles with spin 1/2 and a photon gas was calculated depending on the number of occupied microstates N. For a numerically computed number $N = 137.1356..$, an equality of the entropies of electrons to the entropy of an according photon gas was found. This result is discussed with respect to Sommerfelds fine-structure constant.

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PACS 44.40.+a – Thermal radiation.
PACS 45.20.dh – Energy conservation.
PACS 14.60.Cd – Electrons (including positrons).

1. – Introduction

Richard P. Feynman uniquely described the question of the origin of Sommerfelds fine-structure constant, as given in [1]: *"There is a most profound and beautiful question associated with the observed coupling constant, γ, the amplitude for a real electron to emit or absorb a real photon. It is a simple number that has been experimentally determined to be close to* -0.08542455. *(My physicist friends won't recognize this number, because they like to remember it as the inverse of its square: about* 137.03597 *with about an uncertainty of about* 2 *in the last decimal place. It has been a mystery ever since it was discovered more than fifty years ago, and all good theoretical physicists put this number up on their wall and worry about it.) Immediately you would like to know where this number for a coupling comes from: is it related to pi or perhaps to the base of natural logarithms? Nobody knows. It's one of the greatest damn mysteries of physics: a magic*

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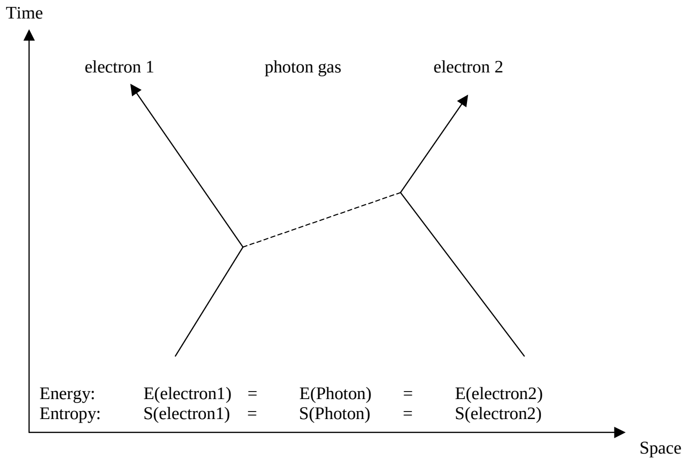

Fig. 1. – Feynman graph of the electromagnetic interaction between two electrons and transport of energy and entropy. The energy of electron 1 has to be transferred to the energy of the photon and then to the energy of electron 2, as well as the entropy, which has to be transferred from electron 1 to the interaction photon and then to electron 2.

number that comes to us with no understanding by man. You might say the "hand of God" wrote that number, and "we don't know how He pushed his pencil". We know what kind of a dance to do experimentally to measure this number very accurately, but we don't know what kind of dance to do on the computer to make this number come out, without putting it in secretly!".

Sommerfelds fine-structure constant is obtained by dividing the constant of the electromagnetic interaction $e^2/4\pi\varepsilon_0$ by the Heisenberg constant ($\hbar \approx 1.0546 \cdot 10^{-34}$ J·s) and the speed of light in vacuum ($c \approx 2.9979 \cdot 10^8$ m/s):

$$\alpha = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{\hbar c} \approx \frac{1}{137.035\ldots}\,, \tag{1}$$

where e stands for the elementary charge ($1.602 \cdot 10^{-19}$ A·s), and ε_0 ($8.845 \cdot 10^{-12}$ A·s·m/V) for the dielectric constant of the vacuum. It originated from calculating the speed in units of the speed of light of an electron circling at the ground state of the hydrogen atom. All four natural constants e, ε_0, c and $\hbar$ can be measured in the laboratory with a sufficient accuracy and from various and independent experiments. The numerical values of these natural constants depend on the system of units. The values given are valid for the International System of Units (SI). In another system of units, other values for e, ε_0, c and $\hbar$ could result. But the fine-structure constant α itself has no unit and is equal for all systems of units. It has the same numerical value in every system and is independent of the chosen system of units. So the fine-structure constant is a natural constant indeed.

The electromagnetic interaction of two electrons can be depicted by Feynman graphs (fig. 1). Using the above definition, the electrostatic interaction energy of two electrons can be written in the form

$$E_{\mathrm{elm}} = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{R} = \alpha\frac{\hbar c}{R}\,. \tag{2}$$

Feynman's question about how one could calculate the numerical value of α from simple assumptions, shall be investigated with the following approach, based on a thermodynamical-statistical view of the interaction of electrons and light.

The question for the coupling constant seems to be the question for the nature of light and its interaction with charged particles (electrons). Light, from the point of quantum theory, is considered to consist of bosons, whereas charged particles (electrons) are fermions. The interaction between bosons and fermions seems to take place in such a way that energy and entropy are both conserved. Energy due to the law of the conservation of energy, which results of the time symmetry (Emmy Noether's theorem), and entropy due to the universal second law of thermodynamics $\mathrm{d}S/\mathrm{d}t \geq 0$, in the case of the electromagnetic interaction $\mathrm{d}S = 0$, because of full reversibility. Reversing the direction of time, the above Feynman graph would depict the interaction of positrons, which experimentally react in the very same way as electrons. For the following approach, we ask for the number of micro-states necessary for this interaction due to conservation of energy and entropy.

2. – Theory

The density of entropy of a photon gas is calculated in statistical thermodynamics, with the result [2]

$$\frac{S_{\text{photon}}}{V} = k\frac{8}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \zeta(4) \tag{3}$$

(where $\zeta(4) = \pi^4/90$; k = Boltzmann's constant). For the photon density one finds

$$\frac{N}{V} = \frac{2}{\pi^2}\frac{(kT)^3}{(\hbar c)^3}\zeta(3) \tag{4}$$

(with $\zeta(3) = 1.2020567..$). Both equations can be combined to calculate the entropy, depending only on the number N of photons, which is hereafter interpreted as the number of microstates necessary for the interaction

$$S = k\frac{4\zeta(4)}{\zeta(3)}N. \tag{5}$$

Thus the entropy of the photon gas is simply proportional to the number of photons N. In contrast, the entropy of a particle gas of N *distinguishable* particles is proportional to the logarithm of the faculty of microstates $\ln N!$.

So, questioning at which N both entropies are equal, as for the interaction of photons with classical particles, one would need to solve the following equation ($4\zeta(4)/\zeta(3) = 3.60157072$):

$$\frac{4\zeta(4)}{\zeta(3)}N = \ln N!. \tag{6}$$

This equation has only one non-trivial solution at $N = 96.382..$ (interpolated between the two nearest neighbours). That is interesting! The amount of entropy depends on

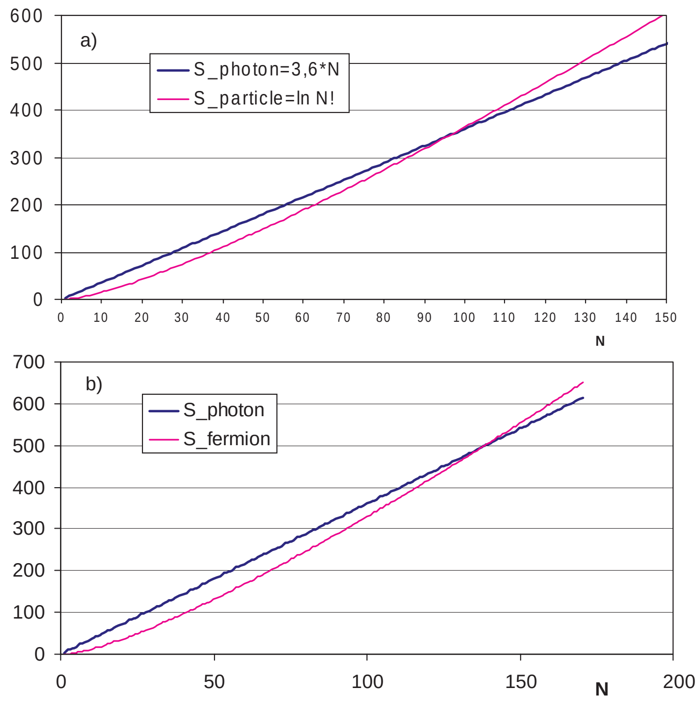

Fig. 2. – Calculation of the interaction entropy for photons, classical particles and fermions with spin 1/2 and estimation of the number N of microstates for the equality of interaction entropy.

if the particles are either *distinguishable* (like classical particles) or if they are *undistinguishable* (like photons). So, the amount of microstates necessary for the interaction between *distinguishable* and *undistinguishable* particles can be calculated via the entropy. Interestingly, as shown in fig. 2, at smaller values of microstates N, the photon regime is dominating the entropy, whereas at larger values of microstates N, the particle-like regime dominates.

Equation (6) considered only classical particles. Now the question about how *distinguishable* quants interact with *undistinguishable* quants, shall be considered. There are exactly two types of quants, according to their symmetry behaviour: Fermi Quants and Bose Quants.

Bosons (photons) are *undistinguishable* and may be created or extinguished arbitrarily. The above equations (3), (4) include this behaviour due to their derivation via Bose statistics.

Fermions are *distinguishable*. For them, a $\ln(N!)$-entropy is valid, but one needs to obey additionally the spin-entropy of the spin-(1/2) electrons. In general, the spin entropy of non-coupling spins (like of ions in a paramagnet) is calculated by ($J = 1/2$ for single electrons)

$$S = -k\ln(2J+1)^{N-1} = -(N-1)k\ln 2. \tag{7}$$

This formula was experimentally verified at Eu^{3+}, Pd^{2+} salts, as described in [2], p. 353ff. The factor $(N-1)$ is reasonable, as for one single electron ($N = 1$), no spin entropy exists.

To calculate the entropy transferred from electron 1 to electron 2 (fig. 1), one would simply add both terms (the $\ln N!$ term for distinguishable particles and the spin-entropy). However, in 50% probability, there has no spin entropy to be transferred from electron 1 via photons to electron 2, as both electrons may be ordered with spin: UP equally (or both electrons with spin: DOWN). Only in the cases where the electrons are ordered with different spin direction (electron 1: UP and electron 2: DOWN; or electron 1: DOWN and electron 2: UP), there has to be transferred spin entropy by the photons, so that one has to take only the half amount of spin entropy into account, yielding for the electron entropy depending on the number N of microstates

$$S = k \ln \left(N! \left(\frac{1}{2}\right)^{(N-1)/2} \right). \tag{8}$$

3. – Results and conclusions

When the entropy equilibrium is calculated, the following equation is to be solved:

$$\frac{4\zeta(4)}{\zeta(3)} N = \ln \left(N! \left(\frac{1}{2}\right)^{(N-1)/2} \right) \tag{9}$$

instead of eq. (6), with its solution of 137.1356131.. (the gamma-function $\Gamma(N+1) = N!$ was used for this calculation).

Using the faculty $N!$ and interpolating linearly between the nearest neighbours, a solution of $N = 137.1278..$ was obtained.

The thermodynamical statistics of microstates N is always possible, also with small numbers N, as the mathematically derivation of the theory of the thermostatistical analysis of microstates starts at small N. For the given manuscript, usual simplifications, as for instance done with the Stirling approximation $\ln N! \approx N \ln N$ for large numbers of microstates, were avoided, as the given manuscript uses the correct formula for $\ln N!$ by using the gamma-function to numerically calculate the faculty values.

However, the entropy as an observable has an average value and its deviation. For instance, due to vacuum fluctuations, there could be no mathematically exact lines drawn in the above entropy/microstates-diagram, but rather smeared stripes, so that the first contact of both lines occurs a little bit earlier. So, one could expect the approach to the experimentally known value of 137.03597 by applying higher degrees of fluctuation corrections.

To give some intuitive comprehension of the connection of the number of microstates towards the photon number and the coupling constant, Planck's assumption $e = nhv$ of the photon energy e at a mode with frequency v and occupation number n might be imagined. In the case of a thermodynamic system with only one mode v, n photons can be present at this mode v. These n photons would occupy the according phase space cell and build up n microstates.

To give a comprehensive explanantion of the origin of the interaction energy of the electromagnetic interaction (eq. (2)), now Heisenberg's uncertainty principle, connecting the uncertainties for energy and distance is considered:

$$\Delta E \Delta x \geq \hbar c\,. \tag{10}$$

In case we have two particles at the distance $R = \Delta x$, the whole system (the particles and the photons of their interaction) is allowed to "borrow" an energy amount of totally ΔE "out of nothing". This energy amount is assumed to be distributed equipartitionally over the N microstates of the thermodynamic system, so $\Delta E \equiv N \cdot E$. One electron itself experiences therefore only the $1/N$-th part of the quantum energy ΔE, as the other $(N-1)/N$ parts are spread to the other $N-1$ microstates, where the questionable electron is recently not to be found. Thus, one might write

$$NE\Delta x = \hbar c\,, \tag{11}$$

$$NER = \hbar c\,, \tag{12}$$

$$E = \frac{\hbar c}{R}\frac{1}{137.1356..}\,. \tag{13}$$

Compared to eq. (2), one might consider the electrostatic energy as a quantum-mechanical energy gained "out of nothing", but being only the 137th part of the energy possible, because the interaction between electrons and photons needs that many microstates take place obeying energy and entropy conservation.

Similar considerations should be made for other types of interactions (gravitation, weak and strong interaction). Having understood an interaction, one should be able to calculate its coupling constant. Understanding gravitation and electromagnetic interaction, one should additionally be able to calculate the lepton masses, for instance. In case the entropy curves should not intersect but had only a point of nearest approach to each other, an amount of irreversible entropy would be produced by this interaction, according to real loss (or brake) of (CP-) symmetry, which could be the case like at the kaon(0) or B(0)-meson decay. So the consideration of entropy could be a fruitful tool for the understanding of interactions.

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[2] Greiner W., *Theoretische Physik*, Vol. **9**, *Thermodynamik und Statistik*, 1. Auflage 1987, p. 334.