

An exact isotropic solution^(*)

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Summary. — The condition for pressure isotropy is reduced to a recurrence equation with variable, rational coefficients of order three. We prove that this difference equation can be solved in general. Consequently we can find an exact solution to the field equations corresponding to a static spherically symmetric gravitational potential in terms of elementary functions. The metric functions, the energy density and the pressure are continuous and well behaved which implies that this solution could be used to model the interior of a relativistic sphere. The model satisfies a barotropic equation of state in general which approximates a polytrope close to the stellar centre.

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1. – Introduction

Static solutions of the Einstein field equations for spherically symmetric manifolds are important in the description of relativistic spheres in astrophysics. The models generated may be used to describe highly compact objects where the gravitational field is strong as in neutron stars. It is for this reason that many investigators use a variety of mathematical techniques to attain exact solutions. One of the first models, satisfying all the physical requirements for a neutron star, was found by Durgal and Bannerji [1]. Now there exist a number of comprehensive collections [2-4] of static, spherically symmetric solutions which provide a useful guide to the literature. It is important to note that only a few of these solutions correspond to nonsingular metric functions with a physically acceptable energy momentum tensor.

In this paper we seek a new exact solution to the field equations which can be used to describe the interior of a relativistic sphere. We rewrite the Einstein equations as a

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new set of differential equations which facilitates the integration process. We choose a cubic form for one of the gravitational potentials, which we believe has not been studied before, which enables us to simplify the condition of pressure isotropy. This yields a third-order recurrence relation, which we manage to solve from first principles. It is then possible to exhibit a new exact solution to the Einstein field equations. The curvature and thermodynamical variables appear to be well behaved. We also demonstrate the existence of an explicit barotropic equation of state. For small values of the radial coordinate close to the stellar core the equation of state approximates a polytrope. We believe that a detailed physical analysis of our solution is likely to lead to a realistic model for compact objects.

2. – Static space-times

Since our intention is to study relativistic stellar objects it seems reasonable, on physical grounds, to assume that space-time is static and spherically symmetric. This is clearly consistent with models utilised to study physical processes in compact objects. The generic line element for static, spherically symmetric space-times is given by

$$(1) \quad ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

in Schwarzschild coordinates. For neutral perfect fluids the Einstein field equations can be written in the form

$$(2a) \quad \frac{1}{r^2} [r(1 - e^{-2\lambda})]' = \rho,$$

$$(2b) \quad -\frac{1}{r^2} (1 - e^{-2\lambda}) + \frac{2\nu'}{r} e^{-2\lambda} = p,$$

$$(2c) \quad e^{-2\lambda} \left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r} \right) = p$$

for the line element (1) where the energy density ρ and the pressure p are measured relative to the comoving fluid 4-velocity $u^a = e^{-\nu}\delta_0^a$ and primes denote differentiation with respect to the radial coordinate r . In the field equations (2) we are using units where the coupling constant $8\pi G/c^4 = 1$ and the speed of light is $c = 1$. An equivalent form of the field equations is obtained if we use the transformation

$$(3) \quad A^2 y^2(x) = e^{2\nu(r)}, \quad Z(x) = e^{-2\lambda(r)}, \quad x = Cr^2$$

due to Durgapal and Bannerji [1], where A and C are arbitrary constants. Under the transformation (3), the system (2) becomes

$$(4a) \quad \frac{1 - Z}{x} - 2\dot{Z} = \frac{\rho}{C},$$

$$(4b) \quad 4Z \frac{\dot{y}}{y} + \frac{Z - 1}{x} = \frac{p}{C},$$

$$(4c) \quad 4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1)y = 0,$$

where the overdot denotes differentiation with respect to the variable x . Note that (4) is a system of three equations in the four unknowns ρ, p, y and Z . The advantage of

this system lies in the fact that a solution can, upon a suitable specification of $Z(x)$, be readily obtained by integrating (4c) which is second order and linear in y .

3. – A new series solution

A large number of exact solutions are known for the system of equations (4) that model a relativistic star with no charge. Many of these are listed by Stephani *et al.* [2] and Finch and Skea [3]. A comprehensive list of static solutions, that satisfy stringent conditions for spherically symmetric perfect fluids, was compiled by Delgaty and Lake [4]. The Einstein field equations in the form (4) are under-determined. From inspection it is clear that the simplest solutions to the system (4) correspond to polynomial forms for $Z(x)$. As far as we are aware all exact solutions found previously correspond to forms of the gravitational potential $Z(x)$ which are linear or quadratic in the independent variable x . Our approach here is to specify the gravitational potential $Z(x)$ and attempt to solve (4c) for the potential y . In an attempt to obtain a new solution to the system (4) we make the choice

$$(5) \quad Z = ax^3 + 1,$$

where a is a constant. We suspect that the cubic form (5) has not been considered before because the resulting differential equation in the dependent variable y is difficult to solve; quadratic forms for Z are listed by Delgaty and Lake [4]. The quadratic form for the potential Z is simpler to handle and contains the familiar Tolman models. With the specified function Z , the condition of pressure isotropy (4c) becomes

$$(6) \quad 2(ax^3 + 1)\ddot{y} + 3ax^2\dot{y} + axy = 0.$$

The linear second-order differential equation (6) is difficult to solve when $a \neq 0$. We have not found a solution for $a \neq 0$ in standard handbooks of differential equations. Software packages such as Mathematica have also not been helpful as they generate a solution in terms of hypergeometric functions with complex arguments. We attempt to find a series solution to (6) using the method of Frobenius. As the point $x = 0$ is a regular point of (6), there exist two linearly independent solutions of the form of a power series with centre $x = 0$. We therefore can write

$$(7) \quad y(x) = \sum_{n=0}^{\infty} c_n x^n,$$

where the c_n are the coefficients of the series. For a legitimate solution we need to determine the coefficients c_n explicitly.

Substituting (7) into (6) yields

$$4c_2 + (12c_3 + ac_0)x + 4(6c_4 + ac_1)x^2 + \sum_{n=2}^{\infty} \{a[2n^2 + n + 1]c_n + 2(n+3)(n+2)c_{n+3}\}x^{n+1} = 0.$$

For this equation to hold true for all x we require

$$(8a) \quad 4c_2 = 0,$$

$$\begin{aligned}
(8b) \quad & 12c_3 + ac_0 = 0, \\
(8c) \quad & 6c_4 + ac_1 = 0, \\
(8d) \quad & a[2n^2 + n + 1]c_n + 2(n+3)(n+2)c_{n+3} = 0, \quad n \geq 2.
\end{aligned}$$

Equation (8d) is a linear recurrence relation with variable, rational coefficients of order three. General techniques of solution for difference equations are limited to the simplest cases and (8d) does not fall into the known classes. However it is possible to solve (8d) from first principles. Equations (8a) and (8d) imply

$$(9) \quad c_2 = c_5 = c_8 = \dots = 0.$$

From (8b) and (8d) we generate the expressions

$$\begin{aligned}
c_3 &= -\frac{a}{2} \frac{1}{3.2} c_0, \\
c_6 &= \frac{a^2}{2^2} \frac{2.3^2 + 3 + 1}{6.3} \frac{1}{5.2} c_0, \\
c_9 &= -\frac{a^3}{2^3} \frac{2.6^2 + 6 + 1}{9.6.3} \frac{2.3^2 + 3 + 1}{8.5.2} c_0.
\end{aligned}$$

It is clear that the coefficients c_3, c_6, c_9, \dots can all be written in terms of the coefficient c_0 . These coefficients generate a pattern and we can write

$$(10) \quad c_{3n+3} = (-1)^{n+1} \left(\frac{a}{2}\right)^{n+1} \prod_{k=0}^n \frac{2(3k)^2 + 3k + 1}{(3k+3)(3k+2)} c_0,$$

where we have utilised the conventional symbol \prod to denote multiplication. We can obtain a similar formula for the coefficients c_4, c_7, c_{10}, \dots . From (8c) and (8d) we have

$$\begin{aligned}
c_4 &= -\frac{a}{2} \frac{2.1^2 + 1 + 1}{4.3} c_1, \\
c_7 &= \frac{a^2}{2^2} \frac{2.4^2 + 4 + 1}{7.4} \frac{2.1^2 + 1 + 1}{6.3} c_1, \\
c_{10} &= -\frac{a^3}{2^3} \frac{2.7^2 + 7 + 1}{10.7.4} \frac{2.4^2 + 4 + 1}{9.6.3} \frac{2.1^2 + 1 + 1}{1} c_1.
\end{aligned}$$

The coefficients c_4, c_7, c_{10}, \dots can all be written in terms of the coefficient c_1 . These coefficients generate a pattern which is clearly of the form

$$(11) \quad c_{3n+4} = (-1)^{n+1} \left(\frac{a}{2}\right)^{n+1} \prod_{k=0}^n \frac{2(3k+1)^2 + (3k+1) + 1}{(3k+4)(3k+3)} c_1,$$

where \prod denotes multiplication.

From (9) we observe that the coefficients c_2, c_5, c_8, \dots all vanish. The coefficients c_3, c_6, c_9, \dots are generated from (10). The coefficients c_4, c_7, c_{10}, \dots are generated from

(11). Hence the difference equation (8d) has been solved and all nonzero coefficients are expressible in terms of the leading coefficients c_0 and c_1 . We can write the series (7) as

$$\begin{aligned}
(12) \quad y(x) &= c_0 + c_1x^1 + c_3x^3 + c_4x^4 + c_6x^6 + c_7x^7 + c_9x^9 + c_{10}x^{10} + \dots \\
&= c_0 \left(1 + \sum_{n=0}^{\infty} c_{3n+3}x^{3n+3} \right) + c_1 \left(x + \sum_{n=0}^{\infty} c_{3n+4}x^{3n+4} \right) \\
&= c_0 \left(1 + \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{a}{2} \right)^{n+1} \prod_{k=0}^n \frac{2(3k)^2 + 3k + 1}{(3k+3)(3k+2)} x^{3n+3} \right) + \\
&\quad + c_1 \left(x + \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{a}{2} \right)^{n+1} \prod_{k=0}^n \frac{2(3k+1)^2 + (3k+1) + 1}{(3k+4)(3k+3)} x^{3n+4} \right),
\end{aligned}$$

where c_0 and c_1 are arbitrary constants. Clearly (12) is of the form

$$(13) \quad y(x) = c_0 y_1(x) + c_1 y_2(x),$$

where

$$\begin{aligned}
y_1(x) &= \left(1 + \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{a}{2} \right)^{n+1} \prod_{k=0}^n \frac{2(3k)^2 + 3k + 1}{(3k+3)(3k+2)} x^{3n+3} \right), \\
y_2(x) &= \left(x + \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{a}{2} \right)^{n+1} \prod_{k=0}^n \frac{2(3k+1)^2 + (3k+1) + 1}{(3k+4)(3k+3)} x^{3n+4} \right)
\end{aligned}$$

are linearly independent solutions of (6). Therefore we have found the general solution to the differential equation (6) for the particular gravitational potential Z given in (5). The advantage of the solutions in (13) is that they are expressed in terms of a series with real arguments unlike the complex arguments given by software packages.

4. – Physical models

From (13) and the Einstein field equations (4) we generate the exact solution

$$(14a) \quad e^{2\lambda} = \frac{1}{ax^3 + 1},$$

$$(14b) \quad e^{2\nu} = A^2 y^2,$$

$$(14c) \quad \frac{\rho}{C} = -7ax^2,$$

$$(14d) \quad \frac{p}{C} = 4(ax^3 + 1) \frac{\dot{y}}{y} + ax^2.$$

In the above the quantity y is given by (13), $x = Cr^2$ and a is a constant. This solution has a simple form and is expressed completely in terms of elementary functions. The expressions given above have the advantage of simplifying the analysis of the physical features of the solution, and will assist in the description of relativistic compact bodies such as neutron stars.

Consider a relativistic sphere where $0 \leq x \leq R$ where $b = CR^2$ and R is the stellar radius. We note that the functions ν and λ have constant values at the centre $x = 0$. The function ρ vanishes at the centre. The pressure p has a constant value at $x = 0$. Hence the gravitational potentials and the matter variables are finite at the centre. Since $y(x) = c_0 y_1(x) + c_1 y_2(x)$ is a well-defined series on the interval $[0, b]$ the quantities ν , λ , ρ and p are nonsingular and continuous. If $a < 0$, then the energy density $\rho > 0$. The constants c_0 and c_1 can be chosen such that the pressure $p > 0$. Consequently the energy density and the pressure are positive on the interval $[0, b]$. At the boundary $x = b$ we must have

$$e^{-2\lambda(R)} = aC^3 R^6 + 1 = 1 - \frac{2M}{R}$$

for a sphere of mass M ; this ensures that the interior space-time matches smoothly to the Schwarzschild exterior. For the speed of sound to be less than the speed of light we require that

$$0 \leq \frac{dp}{d\rho} \leq 1$$

in our units. This inequality will constrain the values of the constants a , c_0 , c_1 , A and C . From this qualitative analysis we believe that the solution found can be used as a basis to describe realistic relativistic stars. We believe that a detailed physical analysis is likely to lead to realistic models for compact objects.

Our solution has the interesting feature of admitting an explicit barotropic equation of state. We observe from (14c) that

$$x = \sqrt{\frac{\rho}{-7aC}}, \quad a < 0$$

and the variable x can be written in terms of ρ only. The function y in (13) can be expressed in terms of ρ and the variable x is eliminated. Consequently the pressure p in (14d) is expressible in terms of ρ only, and we can write

$$p = p(\rho).$$

Thus the solution in (14) obeys a barotropic equation of state. This highly desirable feature is unusual for most exact solutions as pointed out in Stephani *et al.* [2]. For small values of x close to the stellar centre we have $y \approx c_0 + c_1 x$. Then from (14c) and (14d) we have the approximation

$$(15) \quad \frac{p}{C} \approx \frac{4c_1}{c_0 + c_1 \sqrt{\frac{\rho}{-7aC}}}.$$

Therefore for small values of x close to stellar centre (15) implies that we have the approximate equation of state

$$p \propto \rho^{-1/2},$$

which is of the form of a polytrope.

We point out that the solutions presented in this paper may be extended to anisotropic matter. In recent years a number of researchers have proposed models corresponding to anisotropic matter where the radial component of the pressure differs from the angular component. The physical motivation for the analysis of anisotropic matter is that anisotropy affects the critical mass, critical surface redshift and stability of highly compact bodies. These investigations are contained in the papers [5-12], among others. It appears that anisotropy may be important in fully understanding the gravitational behaviour of boson stars and the role of strange matter with densities higher than neutron stars. Mak and Harko [11] and Sharma and Mukherjee [13] have observed that anisotropy is a crucial ingredient in the analysis of dense stars with strange matter. The simple form of our solutions allows for extension to study such matter by adapting the energy momentum tensor to include both radial and tangential pressures.

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REFERENCES

- [1] DURGAPAL M.C. and BANNERJI R., *Phys. Rev. D*, **27** (1983) 328.
- [2] STEPHANI H., KRAMER D., MACCALLUM M. A. H., HOENSELAERS C. and HERLT E., *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge) 2003.
- [3] FINCH M. R. and SKEA J., <http://edradour.symbcomp.uerj.br/> 1998.
- [4] DELGATY M. S. R. and LAKE K., *Comput. Phys Commun.*, **115** (1998) 395.
- [5] CHAISI M. and MAHARAJ S. D., *Gen. Rel. Grav.*, **37** (2005) 117.
- [6] DEV K. and GLEISER M., *Gen. Rel. Grav.*, **34** (2002) 1793.
- [7] DEV K. and GLEISER M., *Gen. Rel. Grav.*, **35** (2003) 1435.
- [8] HERRERA L., MARTIN J. and OSPINO J., *J. Math. Phys.*, **43** (2002) 4889.
- [9] HERRERA L., PRISCO A. D., MARTIN J., OSPINO J., SANTOS N. O. and TROCONIS O., *Phys. Rev. D*, **69** (2004) 084026.
- [10] IVANOV B. V., *Phys. Rev. D*, **65** (2002) 10411.
- [11] MAK M. K. and HARKO T., *Chin. J. Astron. Astrophys.*, **2** (2002) 248.
- [12] MAK M. K. and HARKO T., *Proc. R. Soc. London, Ser. A*, **459** (2003) 393.
- [13] SHARMA R. and MUKHERJEE S., *Mod. Phys. Lett. A*, **17** (2002) 2535.