Surface terms, angular momentum and Hamilton-Jacobi formalism(*)

Y. Güler(**), D. Baleanu(***) and M. Cenk(**)

Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences
Cankaya University - 06530 Ankara, Turkey

Summary. — Quadratic Lagrangians are introduced adding surface terms to a free-particle Lagrangian. Geodesic equations are used in the context of the Hamilton-Jacobi formulation of a constrained system. The manifold structure induced by the quadratic Lagrangian is investigated.

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1. – Introduction

There are mainly two basic methods to investigate constrained systems. The first one is the method initiated by Dirac [1]. This method is based upon the classification of constraints and consistency conditions derived from the variations of the constraints. The symplectic structure is established by defining Dirac brackets. The second method is the treatment of the constrained systems by Caratheodory's equivalent Lagrangian method [2-4]. Since this method leads us to the Hamilton-Jacobi equations, it will be called Hamilton-Jacobi (HJ) approach. Equivalent Lagrangian method [2] leads us to a set of Hamilton-Jacobi equations [3, 4]. Despite of many attempts to clarify the many aspects of Hamilton-Jacobi formalism (HJ) we still have some subtle ones. The main difference between Dirac's formulation [1] and HJ approach [5-7] is the fact that the second one is a multi Hamiltonian approach depending on the rank of the Hessian matrix. On the other hand, the boundary conditions [8-11] play an important role for constrained systems [12] and they are crucial for HJ formulation [2, 13]. The presence of

many Hamiltonians, which are constraints having special forms, suggests us to investigate the possibility to describe the superintegrable systems [14] and to obtain some integrable geometries.

In this paper we start with a free-Lagrangian theory and we add a surface term involving the components of the angular momentum. By a suitable choosing of Lagrange multipliers we obtain a quadratic Lagrangian, which can be singular or not. The aim of this paper is to study quadratic, singular Lagrangians, using the HJ formulation. This paper is arranged as follows: In sect. **2** HJ formalism is briefly explained. In sect. **3** the method of obtaining quadratic Lagrangians is presented and two non-singular quadratic systems are studied. In sect. **4** the singular case is analyzed using the HJ formalism. Conclusions are presented in sect. **5**. In the appendix some geometrical properties of the obtained metrics are shown.

2. – Hamilton-Jacobi formalism

Let us assume that the Lagrangian L is singular and the Hessian matrix $\frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}$, $a, b = 1, 2, .., n - r$, has rank $n - r$. The canonical Hamiltonian H_0 is defined as

$$(1) \qquad H_0 = p_i \dot{q}^i - L(t, q^i, \dot{q}^i).$$

The other Hamiltonians are obtained from the definitions $p_\alpha = \frac{\partial L}{\partial q^\alpha} \mid_{\dot{q}_a = \omega_a}$ and they have the forms

$$(2) \qquad H'_\alpha = H_\alpha(t_\beta, q_a, p_a) + p_\alpha,$$

where $\alpha, \beta = n - r + 1, .., n;\ a = 1, 2, .., n - r$. Thus, one arrives at the following set of Hamilton-Jacobi partial differential equations:

$$(3) \qquad \frac{\partial S}{\partial q^\alpha} + H_\alpha \left(t^\beta, q^a, p_a = \frac{\partial S}{\partial q^a} \right) = 0.$$

The following total differential equations lead us to determine the Hamilton-Jacobi function $S(t^\alpha, q^a)$:

$$(4) \qquad \mathrm{d}q^a = \frac{\partial H'_\alpha}{\partial p_a} \mathrm{d}t^\alpha,$$

$$(5) \qquad \mathrm{d}p_a = -\frac{\partial H'_\alpha}{\partial q^a} \mathrm{d}t^\alpha,$$

$$(6) \qquad \mathrm{d}p_\mu = -\frac{\partial H'_\alpha}{\partial t^\mu} \mathrm{d}t^\alpha.$$

Solutions of these equations $S(q_i)$ as

$$(7) \qquad \mathrm{d}z = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) \mathrm{d}t^\alpha,$$

where $z = S(t^\alpha, q^a)$. Since the equations of motion are total differential equations, one should check the differentiability conditions, consistency conditions in Dirac formulation.

It is proved that the system of eqs. (4) is differentiable if and only if the following Poisson brackets equations are satisfied:

$$[H'_\alpha, H'_\beta] = 0. \tag{8}$$

In general some of these Poisson brackets may not vanish identically. In such cases one should include the new functions as "new" constraints. Poisson brackets of these constraints with H'_α should be considered also. This procedure should continue until all Poisson brackets vanish identically. This approach was studied in detail in refs. [3, 4].

3. – The method

Let us assume that a given Lagrangian $L(\dot{q}^i, q^i) = (1/2)a_{ij}\dot{q}^i\dot{q}^j$ admits a set of independent constants of motion denoted by $L_i, i = 1, \cdots n$. Our aim is to treat the new Lagrangian $L' = L + \dot{\lambda}^i L_i$ as a quadratic Lagrangian expresses as

$$L' = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j. \tag{9}$$

Let us assume that the matrix g_{ij} is non-singular.

Treating g_{ij} as the "metric tensor" of a manifold the geodesic equations

$$\ddot{q}^a = \frac{-1}{2}g^{ab}\left(\frac{\partial g_{bm}}{\partial q^k} + \frac{\partial g_{bk}}{\partial q^m} - \frac{\partial g_{mk}}{\partial q^b}\right)\dot{q}^k\dot{q}^m \tag{10}$$

are equivalent to the Euler-Lagrange equations of (9). Here g^{ab} is the inverse of g_{ab}. In this formulation the canonical Hamiltonian of the system is expressed as $H = (1/2)g^{ab}p_a p_b$. In this way the motion of a particle is described by geodesic equations of a Riemannian manifold with metric tensor g_{ij}. The above motion is treated inside of HJ formalism and the geometrical properties of the obtained metrics g_{ij} will be investigated. In the following we will add to a free Lagrangian the components of the angular momentum.

1) As the first example of a non-singular quadratic system, let us consider the following Lagrangian:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \dot{\lambda}_3(x\dot{y} - y\dot{x}), \tag{11}$$

which can be expressed as $L = (1/2)a_{ij}\dot{q}^i\dot{q}^j$, where

$$a_{ij} = \begin{pmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ -y & x & 0 \end{pmatrix}. \tag{12}$$

This Lagrangian corresponds to the motion of a particle moving on a plane such that L_z is constant.

The Hamiltonian has the following form:

$$H_c = \frac{1}{2}a_{\mu\rho}^{-1}p_\mu p_\rho, \quad \text{where} \quad \mu, \rho = 1, .., 3 \tag{13}$$

and $a_{\mu\rho}^{-1}$ is the inverse of $a_{\mu\rho}$. From the Hamiltonian we find

$$\mathrm{d}x = a_{1\rho}^{-1} p_\rho \mathrm{d}t, \quad \mathrm{d}y = a_{2\rho}^{-1} p_\rho \mathrm{d}t, \quad \mathrm{d}\lambda_3 = a_{3\rho}^{-1} \mathrm{d}t,$$

(14)
$$\mathrm{d}p_x = \frac{-1}{2}\left(\frac{\partial a_{\mu\rho}^{-1}}{\partial x}\right) p_\mu p_\rho \mathrm{d}t, \quad \mathrm{d}p_y = \frac{-1}{2}\left(\frac{\partial a_{\mu\rho}^{-1}}{\partial y}\right) p_\mu p_\rho \mathrm{d}t, \quad \mathrm{d}p_{\lambda_3} = 0.$$

In order to find the equation of motion, we must solve (14). However, we will prefer geodesic equations that produce easier equations and equivalent to (14). The geodesic equations corresponding to (12) are

(15)
$$\ddot{x} = \frac{2x\dot{\lambda}_3}{x^2 + y^2}(x\dot{y} - y\dot{x}),$$
$$\ddot{y} = \frac{2y\dot{\lambda}_3}{x^2 + y^2}(x\dot{y} - y\dot{x}),$$
$$\ddot{\lambda}_3 = \frac{-2\dot{\lambda}_3}{x^2 + y^2}(x\dot{x} + y\dot{y}).$$

The aim is to solve the system of equations given by (15). Notice that (15) implies that

(16)
$$\frac{\ddot{\lambda}_3}{\dot{\lambda}_3} = \frac{-2(x\dot{x} + y\dot{y})}{x^2 + y^2},$$

which admits solutions as

(17)
$$\lambda_3 = \int \frac{C}{x^2 + y^2} \mathrm{d}t,$$

where C is constant. Taking into account that

$$\frac{\ddot{x}}{\ddot{y}} = \frac{x}{y} \quad \text{and} \quad \text{denoting} \quad \frac{\ddot{x}}{x} = \frac{\ddot{y}}{y} = k,$$

we divide our problem into two parts. The first one corresponds to the case when $k < 0$. We obtain the solutions as

(18) $\qquad x(t) = c_1 \sin(\sqrt{k}t) + c_2 \cos(\sqrt{k}t), \quad y(t) = c_3 \sin(\sqrt{k}t) + c_4 \cos(\sqrt{k}t).$

If we impose some initial conditions, one can find specific solutions of equations. Let us assume that $c_1 = 0$, $c_2 = 1$, $c_3 = 1$ and $c_4 = 0$. From (18) we find that $x(t) = \cos(\omega t)$ and $y(t) = \sin(\omega t)$, where $\omega = \sqrt{k}$. Also from eq. (17) we obtain that $\lambda_3 = Ct + B$, where B is a constant. On the other hand, p_x, p_y and p_{λ_3} are given as follows:

(19)
$$\begin{pmatrix} p_x \\ p_y \\ p_{\lambda_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ -y & x & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\lambda}_3 \end{pmatrix}.$$

If we solve (19), we find

(20) $\qquad p_x = \sin(\omega t)(-\omega - C), \quad p_y = \cos(\omega t)(\omega + C), \quad p_{\lambda_3} = \omega.$

In the case of $k > 0$ the solutions are given by

$$(21) \qquad x(t) = c_1 e^{\sqrt{k}t} + c_2 e^{-\sqrt{k}t}, \quad y(t) = c_3 e^{\sqrt{k}t} + c_4 e^{-\sqrt{k}t},$$

where $c_i, i = 1, .., 4$ are constants. As in the previous case λ_3 is given by (17). Using (21) and imposing $A = c_3 c_2 - c_1 c_4 \neq 0$ we obtain the following equation: $(c_4^2 + c_3^2)x^2 + (c_1^2 + c_2^2)y^2 - 2xy(c_2 c_4 + c_1 c_3) = A^2$.

The metric from (12) is conformal flat having its Ricci scalar $R = 2/(x^2 + y^2)$.

2) Let us add now two components of the angular momentum to the Lagrangian of a free three-dimensional particle. We obtain

$$(22) \qquad L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \dot{\lambda}_1(y\dot{z} - z\dot{y}) + \dot{\lambda}_2(z\dot{x} - x\dot{z}),$$

and from (22) we identify the metric a_{ij} as

$$(23) \qquad a_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & z \\ 0 & 1 & 0 & -z & 0 \\ 0 & 0 & 1 & y & -x \\ 0 & -z & y & 0 & 0 \\ z & 0 & -x & 0 & 0 \end{pmatrix}.$$

In this case the geodesic equations have the following forms:

$$(24) \qquad \ddot{x} = \frac{x}{x^2 + y^2 + z^2}[2\dot{\lambda}_1(y\dot{z} - z\dot{y}) + 2\dot{\lambda}_2(z\dot{x} - x\dot{z})],$$

$$(25) \qquad \ddot{y} = \frac{y}{x^2 + y^2 + z^2}[2\dot{\lambda}_1(y\dot{z} - z\dot{y}) + 2\dot{\lambda}_2(z\dot{x} - x\dot{z})],$$

$$(26) \qquad \ddot{z} = \frac{z}{x^2 + y^2 + z^2}[2\dot{\lambda}_1(y\dot{z} - z\dot{y}) + 2\dot{\lambda}_2(z\dot{x} - x\dot{z})],$$

$$(27) \qquad \ddot{\lambda}_1 = \frac{2}{x^2 + y^2 + z^2}\left[-\dot{\lambda}_1\left(\frac{x^2\dot{z} + z^2\dot{z}}{z} + y\dot{y}\right) + \dot{\lambda}_2 y\left(\frac{z\dot{x} - x\dot{z}}{z}\right)\right],$$

$$(28) \qquad \ddot{\lambda}_2 = \frac{2}{x^2 + y^2 + z^2}\left[x\dot{\lambda}_1\left(\frac{z\dot{y} - y\dot{z}}{z}\right) - \dot{\lambda}_2\left(\frac{y^2\dot{z} + z^2\dot{z}}{z} + x\dot{x}\right)\right].$$

These equations form a system of non-linear ordinary differential equation and to solve this system we should notice that $\ddot{x}/x = \ddot{y}/y = \ddot{z}/z$. Assuming that the above ratio is a negative constant k' we obtain

$$(29) \qquad x(t) = c_1 \sin(\sqrt{k}t) + c_2 \cos(\sqrt{k}t),$$

$$(30) \qquad\qquad y(t) = c_3 \sin(\sqrt{k}t) + c_4 \cos(\sqrt{k}t),$$

$$(31) \qquad\qquad z(t) = c_5 \sin(\sqrt{k}t) + c_6 \cos(\sqrt{k}t),$$

where $-k' = k$ and the constants c_i, $i = 1...4$ are subjected to the restriction that $\frac{2\lambda_1(y\dot z - z\dot y) + 2\lambda_2(z\dot x - x\dot z)}{x^2 + y^2 + z^2}$ is a constant. Note that if $k = 0$, both $\ddot x$ and $\ddot y$ must be 0, then x and y are both linear functions of t. λ_1 and λ_2 can be easily obtained from (27) and (28). For the general case the expressions involved are very complicated. Instead of doing this we will present a particular solution. If we assume that $c_2 = \sqrt{2}/2$, $c_3 = 1$, $c_6 = \sqrt{2}/2$ and others are 0, we obtain

$$(32) \qquad x(t) = \frac{\sqrt{2}}{2} \cos(\sqrt{k}t), \quad y(t) = \sin(\sqrt{k}t) \quad \text{and} \quad z(t) = \frac{\sqrt{2}}{2} \cos(\sqrt{k}t).$$

Using (32), (27) and (28) we obtain

$$(33) \qquad\qquad \ddot\lambda_1(t) = 0,$$

$$(34) \qquad\qquad \ddot\lambda_1(t) = \sqrt{2k}\dot\lambda_1(t) - \frac{2\sin(\sqrt{k}t)\sqrt{k}}{\cos(\sqrt{k}t)}\dot\lambda_2(t).$$

The solution of (33) is

$$\lambda_1(t) = k_3 t + k_4,$$

$$(35) \qquad \begin{aligned} \lambda_2(t) &= \frac{1}{4}k_3 \ln(1 + \tan^2(\sqrt{k}t))\sqrt{2}\sqrt{\frac{1}{k}} + \frac{1}{2}k_3 \ln(\cos(\sqrt{k}t))\sqrt{2}\sqrt{\frac{1}{k}} + \\ &\quad + \frac{1}{4\sqrt{k}}k_1 \sin(2\sqrt{k}t) + \left(\frac{\sqrt{2}}{4\sqrt{k}} - \frac{\sqrt{2}}{4k}\cos(2\sqrt{k}t)\right)k_3 + k_2 + \frac{1}{2}k_1 t, \end{aligned}$$

where $k_i, i = 1, \cdots 4$ are constants. As before, p_i have the forms $p_i = a_{ij}\dot q^j$.

4. – The singular case

As an example of a singular quadratic system, let us consider the Lagrangian given by

$$(36) \qquad L = \frac{1}{2}(\dot x^2 + \dot y^2 + \dot z^2) + \dot\lambda_1(y\dot z - z\dot y) + \dot\lambda_2(z\dot x - x\dot z) + \dot\lambda_3(x\dot y - y\dot x),$$

which is expressed in compact form as $L = (1/2)a_{ij}\dot q^i\dot q^j$, where the matrix a_{ij} has the form

$$(37) \qquad a_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & z & -y \\ 0 & 1 & 0 & -z & 0 & x \\ 0 & 0 & 1 & y & -x & 0 \\ 0 & -z & y & 0 & 0 & 0 \\ z & 0 & -x & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the rank of the matrix is 5, this is a singular system. By using $p_i = \partial L/\partial \dot{q}^i$ we find

$$(38) \qquad p_x + y\dot{\lambda}_3 = \dot{x} + z\dot{\lambda}_2, \quad p_y - x\dot{\lambda}_3 = \dot{y} - z\dot{\lambda}_1, \quad p_z = a_{3i}\dot{q}^i,$$
$$p_{\lambda_1} = a_{4i}\dot{q}^i, \quad p_{\lambda_2} = a_{4i}\dot{q}^i.$$

From (38) we find

$$(39)$$
$$\dot{x} = \frac{p_x z x^2 + p_{\lambda_2} z^2 + p_{\lambda_2} y^2 + x p_z z^2 + xy p_{\lambda_1} + xyz p_y}{z(x^2 + y^2 + z^2)},$$
$$\dot{y} = \frac{p_y z y^2 + p_z z^2 y + p_{\lambda_1} z^2 + xyz p_x - xy p_{\lambda_2} - x^2 p_{\lambda_1}}{z(x^2 + y^2 + z^2)},$$
$$\dot{z} = \frac{p_z z^2 + p_y y z + p_{\lambda_1} y - x\lambda_2 - xz p_x}{z(x^2 + y^2 + z^2)},$$
$$\dot{\lambda}_1 = \frac{-p_y z^3 + z^3 x\dot{\lambda}_3 + p_z z^2 y - p_{\lambda_1} z^2 + xyz p_x + x^3 z\dot{\lambda}_3 - zx^2 p_y + zx\dot{\lambda}_3 y^2 - xy p_{\lambda_2} - x^2 p_{\lambda_1}}{z(x^2+y^2+z^2)},$$
$$\dot{\lambda}_2 = \frac{-p_{\lambda_2} z^2 - p_{\lambda_2} y^2 z^3 p_x + z y^2 p_x + z^3 y\dot{\lambda}_3 + y^3 z\dot{\lambda}_3 - p_z x z^2 - p_{\lambda_1} xy - p_y zyx + x^2 yz\dot{\lambda}_3}{z(x^2+y^2+z^2)}.$$

Using (39), the canonical Hamiltonian has the following form:

$$(40) \qquad H_c = \frac{1}{2}b_{\mu\rho}^{-1}p_\mu p_\rho + \dot{\lambda}_3\left(p_{\lambda_3} + \frac{yp_{\lambda_2} + xp_{\lambda_1}}{z}\right) \quad \text{where} \quad \mu, \rho = 1, .., 5,$$

where $b_{\mu\rho}^{-1}$ is the inverse of

$$(41) \qquad b_{\mu\rho} = \begin{pmatrix} 1 & 0 & 0 & 0 & z \\ 0 & 1 & 0 & -z & 0 \\ 0 & 0 & 1 & y & -x \\ 0 & -z & y & 0 & 0 \\ z & 0 & -x & 0 & 0 \end{pmatrix}.$$

In HJ formalism the Hamiltonians to start with are

$$(42) \qquad H_0' = p_0 + \frac{1}{2}b_{\mu\rho}^{-1}p_\mu p_\rho, \quad H_1' = p_{\lambda_3} + \frac{yp_{\lambda_2} + xp_{\lambda_1}}{z}.$$

The total differential equations corresponding to (42) are

$$(43) \qquad dx = b_{1\rho}^{-1}p_\rho dt, \quad dy = b_{2\rho}^{-1}p_\rho dt, \quad dz = b_{3\rho}^{-1}p_\rho dt,$$

$$(44) \qquad d\lambda_1 = b_{4\rho}^{-1}p_\rho dt + \frac{x}{z}d\lambda_3, \quad d\lambda_2 = b_{5\rho}^{-1}dt + \frac{y}{z}d\lambda_3,$$

$$(45) \qquad dp_{\lambda_1} = 0, \quad dp_{\lambda_2} = 0, \quad dp_x = \frac{-1}{2}\left(\frac{\partial b_{\mu\rho}^{-1}}{\partial x}\right)p_\mu p_\rho dt - \frac{p_{\lambda_1}}{z}d\lambda_3,$$

$$(46) \qquad dp_y = \frac{-1}{2}\left(\frac{\partial b_{\mu\rho}^{-1}}{\partial y}\right)p_\mu p_\rho dt - \frac{p_{\lambda_2}}{z}d\lambda_3,$$

$$(47) \qquad dp_z = \frac{-1}{2}\left(\frac{\partial b_{\mu\rho}^{-1}}{\partial z}\right)p_\mu p_\rho dt + \left(\frac{yp_{\lambda_2} + xp_{\lambda_2}}{z^2}\right)d\lambda_3.$$

The next step is to find the variations of H'_0 and H'_1. Using the fact that $b^{-1}_{\mu\rho}$ is a symmetric matrix and taking into account the previous equations we obtain after some calculations that both variations are zero. We conclude that the system is integrable and we observe that $H'_1 = z\vec{r}x\vec{L}$ which is identically zero. The geometrical properties of the 5×5 metrics are presented in the appendix.

4˙1. *Curved space generalization*. – Let us start with a free three-dimensional Lagrangian

$$(48) \qquad\qquad L = \frac{1}{2}(\dot{x^2} + \dot{y^2} + \dot{z^2}).$$

If we impose $z^2 = 1 - x^2 - y^2$ and take its time derivative, we obtain $\dot{z} = -\frac{x\dot{x}+y\dot{y}}{\sqrt{u}}$. So, (48) becomes

$$(49) \qquad\qquad L' = g_{ab}\dot{q}^a\dot{q}^b,$$

where

$$(50) \qquad\qquad g_{ab} = \delta_{ab} + \frac{q^a q^b}{u},$$

with $z^2 = u$, $a, b = 1, 2$. The above metric admits three invariants [15]

$$(51) \qquad\qquad L_x = -\sqrt{u}p_y, \quad L_y = \sqrt{u}p_x, \quad L_z = xp_y - yp_x,$$

Adding those three constants of motion to (49) we obtain the extended Lagrangian in a compact form as $L' = (1/2)a_{ij}\dot{q}^i\dot{q}^j$, where a_{ij} has the following form:

$$(52) \qquad a_{ij} = \begin{pmatrix} 1 + \frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}} + \sqrt{u} & -y \\ \frac{xy}{u} & 1 + \frac{y^2}{u} & -\frac{y^2}{\sqrt{u}} - \sqrt{u} & \frac{xy}{\sqrt{u}} & x \\ -\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}} - \sqrt{u} & 0 & 0 & 0 \\ \frac{x^2}{\sqrt{u}} + \sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix}.$$

The determinant a_{ij} is 0, so a_{ij} is a singular symmetric matrix and the rank of it is 4. In order to find the canonical Hamiltonian, we need $\dot{x}$, $\dot{y}$, $\dot{\lambda}_1$ and $\dot{\lambda}_2$ in terms of p_x, p_y, p_{λ_1}, p_{λ_2} and $\dot{\lambda}_3$. Using

$$(53) \qquad\qquad p_i = \frac{\partial L}{\partial \dot{q}^i} = a_{ij}\dot{q}^j \quad \text{where} \quad i, j = 1, .., 5,$$

we obtain

$$\dot{x} = \frac{(-p_{\lambda_2} + p_{\lambda_2} x^2 + xyp_{\lambda_1})\sqrt{1-x^2-y^2}}{1-x^2-y^2},$$

$$\dot{y} = \frac{(p_{\lambda_1} - p_{\lambda_1} y^2 + xyp_{\lambda_2})\sqrt{1-x^2-y^2}}{1-x^2-y^2},$$

(54)
$$\dot{\lambda_1} = \frac{x^3\dot{\lambda_3}+x^3yp_x-x^2p_y+x^2p_yy^2+xy^2\dot{\lambda_3}-x\dot{\lambda_3}+xy\sqrt{1-x^2-y^2}+xy^3p_x}{\sqrt{1-x^2-y^2}(1-x^2-y^2)} -$$
$$- \frac{xyp_x - p_{\lambda_1}\sqrt{1-x^2-y^2}y^2 - 2p_yy^2 + p_{\lambda_1}\sqrt{1-x^2-y^2} + y^4p_y + p_y}{\sqrt{1-x^2-y^2}(1-x^2-y^2)},$$

$$\dot{\lambda_2} = \frac{p_{\lambda_2}\sqrt{1-x^2-y^2}(x^2-1)-p_x+2x^2p_x-y\dot{\lambda_3}+x^2y\dot{\lambda_3}-x^4p_x+y^2p_x-y^2x^2p_x}{\sqrt{1-x^2-y^2}(1-x^2-y^2)} +$$
$$+ \frac{y^3\dot{\lambda_3}-xy^3p_y+xyp_{\lambda_1}\sqrt{1-x^2-y^2}-x^3yp_y+xyp_y}{\sqrt{1-x^2-y^2}(1-x^2-y^2)} \, ;$$

so, the canonical Hamiltonian is given by

$$(55) \qquad H_c = \frac{1}{2}g_{\mu\rho}^{-1}p_\mu p_\rho + \dot{\lambda_3}\left(p_{\lambda_3} + \frac{yp_{\lambda_2}+xp_{\lambda_1}}{\sqrt{1-x^2-y^2}}\right) \quad \text{where} \quad \mu, \rho = 1, .., 4,$$

where

$$(56) \qquad g_{\mu\rho} = \begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}}+\sqrt{u} \\ \frac{xy}{u} & 1+\frac{y^2}{u} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & \frac{xy}{\sqrt{u}} \\ -\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & 0 & 0 \\ \frac{x^2}{\sqrt{u}}+\sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0 \end{pmatrix}.$$

Since the matrix $g_{\mu\rho}$ is singular, we obtain a constrained system. In HJ formalism we have two Hamiltonians

$$(57) \qquad H_0' = p_0 + \frac{1}{2}g_{\mu\rho}^{-1}p_\mu p_\rho, \quad H_1' = p_{\lambda_3} + \frac{yp_{\lambda_2}+xp_{\lambda_1}}{\sqrt{1-x^2-y^2}}.$$

The total differential equations of (57) are given by

$$dx = g_{1\rho}^{-1}p_\rho dt, \quad dy = g_{2\rho}^{-1}p_\rho dt, \quad d\lambda_1 = g_{3\rho}^{-1}dt + \frac{x}{\sqrt{1-x^2-y^2}}d\lambda_3,$$

$$d\lambda_2 = g_{4\rho}^{-1}dt + \frac{y}{\sqrt{1-x^2-y^2}}d\lambda_3,$$

(58)
$$dp_x = \frac{-1}{2}\left(\frac{\partial g_{\mu\rho}^{-1}}{\partial x}\right)p_\mu p_\rho dt - \left(\frac{p_{\lambda_1}}{\sqrt{1-x^2-y^2}} + \frac{(xp_{\lambda_1}+yp_{\lambda_2})x}{(1-x^2-y^2)^{\frac{3}{2}}}\right)d\lambda_3,$$

$$dp_y = \frac{-1}{2}\left(\frac{\partial g_{\mu\rho}^{-1}}{\partial y}\right)p_\mu p_\rho dt - \left(\frac{p_{\lambda_2}}{\sqrt{1-x^2-y^2}} + \frac{(xp_{\lambda_1}+yp_{\lambda_2})y}{(1-x^2-y^2)^{\frac{3}{2}}}\right)d\lambda_3,$$

$$dp_{\lambda_1} = 0, \quad dp_{\lambda_2} = 0.$$

The variations of H_0' and H_1' are 0. Geodesic equations corresponding to $g_{\mu\rho}$ are given by

$$(59) \qquad \ddot{y}u = y(y^2-1)\dot{x}^2 - 2xy^2\dot{x}\dot{y} + y(x^2-1)\dot{y}^2,$$
$$(60) \qquad \ddot{x}u = x(y^2-1)\dot{x}^2 - 2yx^2\dot{x}\dot{y} + x(x^2-1)\dot{y}^2,$$

(61) $\qquad \ddot{\lambda}_1 u = -2x(y^2-1)\dot{x}\dot{\lambda}_1 - 2y(y^2-1)\dot{x}\dot{\lambda}_2 + 2x^2 y\dot{y}\dot{\lambda}_1 + 2xy^2\dot{y}\dot{\lambda}_2,$

(62) $\qquad \ddot{\lambda}_2 u = -2x(x^2-1)\dot{y}\dot{\lambda}_1 - 2y(x^2-1)\dot{y}\dot{\lambda}_2 + 2x^2 y\dot{x}\dot{\lambda}_1 + 2xy^2\dot{x}\dot{\lambda}_2,$

where $u = 1 - x^2 - y^2$. After some calculations we obtain a solution as

$$x(t) = \cos(kt), \quad y(t) = \sin(kt),$$

(63) $\qquad \lambda_1(t) = \dfrac{1}{k} e^{-\frac{1}{2}\pi + kt} \cos(kt)C_4 - \dfrac{1}{k}\cos(kt)C_3 e^{-\frac{1}{2}\pi - kt} + C_1,$

$$\lambda_2(t) = C_2 k - C_3 e^{-\frac{1}{2}\pi - kt}\sin(kt) + \dfrac{C_4 e^{(-\frac{1}{2}\pi + kt)}\sin(kt)}{k}.$$

The geometrical properties of the corresponding metrics are presented in the appendix.

5. – Conclusions

In this paper we illustrated the importance of the surface terms in finding some integrable geometries in three, four and five dimensions. The central role was played by the components of the angular momentum. We added a surface term $\dot{\lambda}_3 L_z$ to a free two-dimensional Lagrangian and we obtain a three-dimensional geometry which is conformal flat and has a non-vanishing scalar curvature. We solved the geodesics equations and we found non-trivial solutions. We repeated the procedure and we investigated the case when two components of the angular momentum were added to a free three-dimensional Lagrangian. In this manner we obtain a five-dimensional metric admitting three Killing vectors. A singular system was obtained if all components of the angular momentum were added. We treated this system within (HJ) formalism and we obtained three types of four-dimensional metrics admitting three isometries. The procedure was extended to the curved space and the two-dimensional sphere was investigated in details.

$$* \; * \; *$$

APPENDIX

In the following the Christoffel components, Ricci scalar and the Weyl components of the obtained metrics are presented.

1) For the singular case we obtain three 5×5 non-singular matrices from (37). Let us denote k as $k = x^2 + y^2 + z^2$. The first metric is given by (41).

The non-zero Christoffel components are given as

$$\Gamma^1_{24} = -\Gamma^1_{15} = -\frac{z}{y}\Gamma^1_{34} = \frac{z}{x}\Gamma^1_{35} = \Gamma^1_{24} - \frac{x}{y}\Gamma^2_{15} = \frac{x}{y}\Gamma^2_{24} = -\frac{zx}{y^2}\Gamma^2_{34} = \frac{z}{y}\Gamma^2_{35} =$$

$$= -\frac{x}{z}\Gamma^3_{15} = \frac{x}{z}\Gamma^3_{24} = -\frac{x}{y}\Gamma^3_{34} = \Gamma^3_{35} = -\frac{zx}{y}\Gamma^4_{15} = \frac{zx}{y}\Gamma^4_{24} = \frac{z^2 x}{x^2+z^2}\Gamma^4_{34} = \frac{z^2}{y}\Gamma^4_{35} =$$

$$= z\Gamma^5_{15} = -z\Gamma^5_{24} = \frac{z}{y}\Gamma^5_{34} = \frac{z^2 x}{z^2+y^2}\Gamma^5_{35} = \frac{zx}{k}.$$

Ricci scalar is given by $R = \dfrac{12}{x^2 + y^2 + z^2}$ and the Weyl tensor components are

$$W_{1212} = -\frac{1}{3}\frac{2x^2 + z^2 + 2y^2}{k^2}, \quad W_{1213} = -\frac{1}{3}\frac{yz}{k^2}, \quad W_{1214} = -\frac{1}{3}\frac{(z^2 + y^2)z}{k^2},$$

$$W_{1423} = -\frac{1}{3}\frac{(z^2 + y^2)x}{k^2}, \quad W_{1215} = W_{1224} = \frac{1}{3}\frac{xyz}{k^2}, \quad W_{1223} = \frac{1}{3}\frac{xz}{k^2},$$

$$W_{1225} = -\frac{1}{3}\frac{z(x^2 + z^2)}{k^2}, \quad W_{2335} = \frac{1}{3}\frac{xyz}{k^2}, \quad W_{1234} = \frac{1}{3}\frac{z^2 x}{k^2}, \quad W_{1234} = \frac{1}{3}\frac{yz^2}{k^2},$$

$$W_{1313} = -\frac{1}{3}\frac{y^2 + 2x^2 + 2z^2}{k^2}, \quad W_{2323} = -\frac{1}{3}\frac{2y^2 + x^2 + 2z^2}{k^2},$$

$$W_{1523} = W_{2324} = \frac{1}{3}\frac{x^2 y}{k^2}, \quad W_{1314} = \frac{1}{3}\frac{(z^2 + y^2)y}{k^2}, \quad W_{1315} = W_{1324} = -\frac{1}{3}\frac{xy^2}{k^2},$$

$$W_{1323} = -\frac{1}{3}\frac{xy}{k^2}, \quad W_{1325} = \frac{1}{3}\frac{(z^2 + x^2)y}{k^2}, \quad W_{1334} = -\frac{1}{3}\frac{xyz}{k^2},$$

$$W_{1335} = -\frac{1}{3}\frac{y^2 z}{k^2}, \quad W_{2334} = \frac{1}{3}\frac{x^2 z}{k^2}, \quad W_{2325} = -\frac{1}{3}\frac{(x^2 + z^2)x}{k^2}.$$

The second metric is given by

$$(\text{A.1}) \qquad b_{\mu\nu}^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & -y \\ 0 & 1 & 0 & -z & x \\ 0 & 0 & 1 & y & 0 \\ 0 & -z & y & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix}.$$

The non-zero Christoffel symbols are given by

$$\Gamma_{15}^1 = \frac{y}{z}\Gamma_{24}^1 = -\frac{y}{x}\Gamma_{25}^1 = -\Gamma_{34}^1 = \frac{x}{y}\Gamma_{15}^2 = \frac{x}{z}\Gamma_{24}^2 = -\Gamma_{25}^2 = -\frac{x}{y}\Gamma_{34}^2 = \frac{x}{z}\Gamma_{34}^3 =$$

$$= \frac{x}{z}\Gamma_{15}^3 = \frac{xy}{z^2}\Gamma_{24}^3 = -\frac{y}{z}\Gamma_{25}^3 = -\frac{xy}{z}\Gamma_{15}^4 = \frac{xy^2}{x+y}\Gamma_{24}^4 = \frac{y^2}{z}\Gamma_{25}^4 = \frac{xy}{z}\Gamma_{34}^4 = y\Gamma_{15}^5 =$$

$$= \frac{y^2}{z}\Gamma_{24}^5 = \frac{xy^2}{z^2 + y^2}\Gamma_{25}^5 = -y\Gamma_{35}^5 = -\frac{x}{z}\Gamma_{34}^3 = \frac{xy}{k};$$

Ricci scalar $R = \dfrac{12}{x^2 + y^2 + z^2}$.

The Weyl tensor components are

$$W_{1212} = -\frac{1}{3}\frac{z^2 + 2x^2 + 2y^2}{k^2}, \quad W_{1213} = -\frac{1}{3}\frac{zy}{k^2}, \quad W_{1214} = -\frac{1}{3}\frac{(z^2 + y^2)z}{k^2},$$

$$W_{1215} = \frac{1}{3}\frac{z^2 x}{k^2}, \quad W_{1223} = \frac{1}{3}\frac{zx}{k^2}, \quad W_{1224} = \frac{1}{3}\frac{xyz}{k^2}, \quad W_{1225} = \frac{1}{3}\frac{z^2 y}{k^2},$$

$$W_{1234} = \frac{1}{3}\frac{z^2 x}{k^2}, \quad W_{1235} = -\frac{1}{3}\frac{(x^2 + y^2)z}{k^2}, \quad W_{1313} = -\frac{1}{3}\frac{2z^2 + 2x^2 + y^2}{k^2},$$

$$W_{1314} = \frac{1}{3}\frac{y(z^2 + y^2)}{k^2}, \quad W_{1315} = -\frac{1}{3}\frac{xyz}{k^2}, \quad W_{1323} = -\frac{1}{3}\frac{xy}{k^2}, \quad W_{1324} = -\frac{1}{3}\frac{y^2 x}{k^2},$$

$$W_{1325} = -\frac{1}{3}\frac{y^2 z}{k^2}, \quad W_{1334} = -\frac{1}{3}\frac{xyz}{k^2}, \quad W_{1335} = \frac{1}{3}\frac{(x^2+y^2)y}{k^2},$$

$$W_{1423} = -\frac{1}{3}\frac{(z^2+y^2)x}{k^2}, \quad W_{1523} = \frac{1}{3}\frac{zx^2}{k^2}, \quad W_{2323} = -\frac{1}{3}\frac{2z^2+2y^2+x^2}{k^2},$$

$$W_{2324} = -\frac{1}{3}\frac{x^2 y}{k^2}, \quad W_{2325} = \frac{1}{3}\frac{xyz}{k^2}, \quad W_{2334} = \frac{1}{3}\frac{zx^2}{k^2}, \quad W_{2335} = -\frac{1}{3}\frac{(x^2+y^2)x}{k^2}.$$

The third metric is

$$(A.2) \qquad b^{(3)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & z & -y \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & -x & 0 \\ z & 0 & -x & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix}.$$

The non-zero Christoffel symbols of the second kind are given as

$$\Gamma^1_{14} = -\frac{z}{y}\Gamma^1_{15} = \frac{z}{x}\Gamma^1_{25} = -\frac{z}{x}\Gamma^1_{34} = \frac{x}{y}\Gamma^2_{14} = -\frac{zx}{y^2}\Gamma^2_{15} = \frac{z}{y}\Gamma^2_{25} = -\frac{z}{y}\Gamma^2_{34} = \frac{x}{z}\Gamma^3_{14} =$$

$$= -\frac{x}{y}\Gamma^3_{15} = \Gamma^3_{25} = -\Gamma^3_{34} = -\frac{zx}{x^2+y^2}\Gamma^4_{14} = -\frac{x}{y}\Gamma^4_{15} = x\Gamma^4_{25} = -x\Gamma^4_{34} = -\frac{x}{y}\Gamma^5_{14} =$$

$$= -\frac{zx^2}{x^2+z^2}\Gamma^5_{15} = -\frac{zx}{y}\Gamma^5_{25} = \frac{zx}{y}\Gamma^5_{34} = -\frac{zx}{k};$$

Ricci scalar $R = \dfrac{12}{x^2+y^2+z^2}$.

The Weyl tensor components are

$$W_{1212} = -\frac{1}{3}\frac{z^2+2x^2+2y^2}{k^2}, \quad W_{1213} = -\frac{1}{3}\frac{zy}{k^2}, \quad W_{1214} = \frac{1}{3}\frac{zyx}{k^2}, \quad W_{1215} = \frac{1}{3}\frac{z^2 x}{k^2},$$

$$W_{1223} = \frac{1}{3}\frac{zx}{k^2}, \quad W_{1224} = \frac{1}{3}\frac{(z^2+x^2)z}{k^2}, \quad W_{1225} = W_{1234} = \frac{1}{3}\frac{z^2 y}{k^2},$$

$$W_{1235} = -\frac{1}{3}\frac{(x^2+y^2)z}{k^2}, \quad W_{1313} = -\frac{1}{3}\frac{2z^2+2x^2+y^2}{k^2}, \quad W_{1314} = -\frac{1}{3}\frac{y^2 x}{k^2},$$

$$W_{1315} = -\frac{1}{3}\frac{xyz}{k^2}, \quad W_{1323} = -\frac{1}{3}\frac{xy}{k^2}, \quad W_{1324} = \frac{1}{3}\frac{(x^2+z^2)y}{k^2}, \quad W_{1325} = -\frac{1}{3}\frac{y^2 z}{k^2},$$

$$W_{1334} = -\frac{1}{3}\frac{zy^2}{k^2}, \quad W_{1335} = \frac{1}{3}\frac{(x^2+y^2)y}{k^2}, \quad W_{1423} = \frac{1}{3}\frac{x^2 y}{k^2}, \quad W_{1523} = \frac{1}{3}\frac{zx^2}{k^2},$$

$$W_{2323} = -\frac{1}{3}\frac{2z^2+2y^2+x^2}{k^2}, \quad W_{2324} = -\frac{1}{3}\frac{(x^2+z^2)x}{k^2},$$

$$W_{2325} = \frac{1}{3}\frac{xyz}{k^2}, \quad W_{2334} = \frac{1}{3}\frac{xyz}{k^2}, \quad W_{2335} = -\frac{1}{3}\frac{(x^2+y^2)x}{k^2}.$$

2) We are repeating the same procedure as above for the curved case. The metric given in (56) has Ricci scalar equal to 0, $W_{1212} = -1/u$ and the Christoffel symbols of the second-kind non-zero components are

$$\Gamma^1_{11} = -\Gamma^3_{13} = \frac{x(1-y^2)}{u}, \quad \Gamma^2_{22} = -\Gamma^4_{24} = \frac{y(1-x^2)}{u}, \quad \Gamma^1_{22} = -\Gamma^4_{23} = \frac{x(1-x^2)}{u},$$

$$\Gamma^4_{14} = \Gamma^3_{24} = -\Gamma^2_{12} = -\frac{xy^2}{u}, \quad -\Gamma^1_{12} = \Gamma^3_{23} = \Gamma^4_{13} = -\frac{x^2y}{u}, \quad \Gamma^2_{11} = -\Gamma^3_{14} = \frac{y(1-y^2)}{u},$$

$$u = 1 - x^2 - y^2.$$

In this case the second metric is given as

$$(A.3) \qquad g^{(2)}_{\mu\nu} = \begin{pmatrix} 1 + \frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & -y \\ \frac{xy}{u} & 1 + \frac{y^2}{u} & -\sqrt{u} - \frac{y^2}{\sqrt{u}} & x \\ -\frac{xy}{\sqrt{u}} & -\sqrt{u} - \frac{y^2}{\sqrt{u}} & 0 & 0 \\ -y & x & 0 & 0 \end{pmatrix};$$

Ricci scalar is $R = 0$, $W_{1212} = -1/u$.

The Christoffel symbols are

$$\Gamma^1_{11} = \frac{x(1-y^2)}{u}, \quad \Gamma^1_{12} = \frac{x^2y}{u}, \quad \Gamma^1_{22} = \frac{x(1-x^2)}{u}, \quad \Gamma^2_{11} = \frac{y(1-y^2)}{u}, \quad \Gamma^2_{12} = \frac{xy^2}{u},$$

$$\Gamma^2_{22} = \frac{y(1-x^2)}{u}, \quad \Gamma^3_{13} = -x, \quad \Gamma^3_{14} = -\sqrt{u}, \quad \Gamma^3_{23} = -\frac{x^2}{y}, \quad \Gamma^3_{24} = \frac{x}{y}\sqrt{u},$$

$$\Gamma^4_{13} = -\frac{x^2}{\sqrt{u}}, \quad \Gamma^4_{14} = x, \quad \Gamma^4_{23} = \frac{x(-1+x^2)}{\sqrt{-u}}, \quad \Gamma^4_{24} = \frac{x^2-1}{y}.$$

The third metric has the form

$$(A.4) \qquad g^{(3)}_{\mu\nu} = \begin{pmatrix} 1 + \frac{x^2}{u} & \frac{xy}{u} & \sqrt{u} + \frac{x^2}{\sqrt{u}} & -y \\ \frac{xy}{u} & 1 + \frac{y^2}{u} & \frac{xy}{\sqrt{u}} & x \\ \sqrt{u} + \frac{x^2}{\sqrt{u}} & \frac{xy}{\sqrt{u}} & 0 & 0 \\ -y & x & 0 & 0 \end{pmatrix};$$

Ricci scalar is $R = 0$, $W_{1212} = -1/u$.

The Christoffel symbols are

$$\Gamma^1_{11} = \frac{x(1-y^2)}{u}, \quad \Gamma^1_{12} = \frac{x^2y}{u}, \quad \Gamma^1_{22} = \frac{x(1-x^2)}{u}, \quad \Gamma^2_{11} = \frac{y(1-y^2)}{u}, \quad \Gamma^2_{12} = \frac{xy^2}{u},$$

$$\Gamma^2_{22} = \frac{y(1-x^2)}{u}, \quad \Gamma^3_{13} = \frac{y^2}{x}, \quad \Gamma^3_{14} = \frac{y}{x}\sqrt{u}, \quad \Gamma^3_{23} = -y, \quad \Gamma^3_{24} = -\sqrt{u},$$

$$\Gamma^4_{13} = \frac{(-y^2+1)y}{x\sqrt{-u}}, \quad \Gamma^4_{14} = \frac{1-y^2}{x}, \quad \Gamma^4_{23} = \frac{y^2}{\sqrt{u}}, \quad \Gamma^4_{24} = y.$$

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