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Schwinger's mechanism in QCD

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Summary. — We calculate the probability density $\mathrm{d}P/(\mathrm{d}V\,\mathrm{d}t)$ for the creation of pairs of particles in a covariantly homogeneous strong QCD field. Calculation is performed in the limit when the field extends over large distance in all space directions. It is believed that the results describe approximately the actual pair production in effective QCD flux tubes with generally moving end-point quarks. The calculation is not restricted to the case when QCD field has a color-diagonal form. Such QCD flux tubes may occur in various processes involving deep inelastic scattering.

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1. – Introduction

It is well known that an intense gauge field can produce pairs of particles. Schwinger [1] showed this for the (Abelian) case of a strong homogeneous electric field, and Batalin, Matinyan and Savvidi [2] for the case of a (covariantly) homogeneous $SU(2)$-"electric" field. In both cases, it was assumed that the field extends over a large (or infinite) volume, and the field was regarded as external (classical), *i.e.* interactions between the produced particles via exchange of field quanta were neglected. Later on, Casher, Neuberger and Nussinov [3] studied pair production in QCD flux tubes, approximating first the field in the tube with a strong external Abelian electric field. Then they partially took into account the non-Abelian nature of the $SU(3)$ color group by involving in their calculation the diagonal Gell-Mann matrices λ_3 and λ_8. Such approximations are believed to give qualitatively correct results as to the pair production. One next step would be to take into account explicitly the non-Abelian structure of the external (covariantly) homogeneous QCD field in its most general case involving also the the nondiagonal Gell-Mann matrices λ_j $(j = 1, 2, 4, \ldots, 8)$, as well as effectively allowing the end-point quarks to

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move (*i.e.* including QCD-"magnetic" field in addition to the QCD-"electric" field). To our knowledge, such a calculation has not been performed in the literature. Such general QCD flux tube configurations may well arise in various deep inelastic processes (for example, see [3] for $e^+e^- \to q\bar{q}$ processes, and [4] for e-p collisions. Therefore, we will calculate, in the limit of large (infinite) space volume, the pair production probability density $dP/(dV\,dt)$ for such general configurations, leaving aside detailed discussion of possible application of obtained formulas to various physical processes.

2. – Calculation of probability density

The probability density per unit volume and per unit time for vacuum not to remain vacuum $\omega = dP/(dV\,dt)$ is (cf. [5])

$$w = \frac{dP}{d^4x} = -\frac{1}{\Omega}\ln|S_0(B)|^2 = -\frac{2}{\Omega}\mathrm{Re}\ln S_0(B)\,, \tag{2.1}$$

where $x = (x^0, x^1, x^2, x^3)$; Ω is the large four-dimensional space-time volume in which the field is present; $S_0(B)$ is the vacuum-to-vacuum amplitude in the presence of an external strong (QCD) gauge field B^μ. $S_0(B)$ can be conveniently expressed in a form involving path integrals

$$S_0(B) \equiv \langle 0_{\mathrm{in}}|S(B)|0_{\mathrm{in}}\rangle = \langle 0_{\mathrm{out}}|0_{\mathrm{in}}\rangle^{(B)} = Z(B)/Z(0)\,, \tag{2.2}$$

where

$$Z(B) = \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi \exp\left[\mathrm{i}\int d^4x[\mathrm{i}\overline{\Psi}(\partial\!\!\!/ + \mathrm{i}m + \mathrm{i}g_s B)\Psi]\right]. \tag{2.3}$$

Here, Ψ is the Grassman variable corresponding to the degrees of freedom of the produced quarks. Specifically, we will consider the case when w is nonzero due to production of quark-antiquark pairs in the general covariantly homogeneous QCD field B^μ extending in a large (or infinite) volume.

In non-Abelian theories, the space and time homogeneity of a gauge field is not a gauge-invariant, but a gauge-covariant property. This means that the antisymmetric field-tensor of such a field $G^\ell_{\mu\nu}$ is x-independent only in particular gauges where it has the form [2](1)

$$G^\ell_{\mu\nu}(\equiv \partial_\nu b^\ell_\mu - \partial_\mu b^\ell_\nu + \mathrm{i}g_s f^{ij\ell} b^i_\mu b^j_\nu) = -F_{\mu\nu}n^\ell\,, \tag{2.4}$$

where $f^{ij\ell}$ are the $SU(3)$ antisymmetric structure constants, g_s is the QCD gauge coupling parameter, and the parameters relevant for the covariant homogeneity are: (n^ℓ) is an x-independent 8-component real unit vector in the adjoint color space ($\Sigma n^\ell n^\ell = 1$), and $F_{\mu\nu}$ is a general antisymmetric x-independent tensor, *i.e.* it has the form identical

(1) Although the authors of [2] discussed in particular only the $SU(2)$ case, their formulas for the general covariantly homogeneous field apply also in the $SU(3)$ case.

to that of classical homogeneous EM field:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -\mathcal{E}^1 & -\mathcal{E}^2 & -\mathcal{E}^3 \\ \mathcal{E}^1 & 0 & -\mathcal{B}^3 & \mathcal{B}^2 \\ \mathcal{E}^2 & \mathcal{B}^3 & 0 & -\mathcal{B}^1 \\ \mathcal{E}^3 & -\mathcal{B}^2 & \mathcal{B}^1 & 0 \end{bmatrix} . \tag{2.5}$$

The corresponding QCD gauge field has then, in a specific gauge, the following form [2]:

$$B_\mu \equiv \frac{1}{2}\lambda^\ell b^\ell_\mu = -\left(\frac{\hat{n}\cdot\vec{\lambda}}{2}\right)\frac{1}{2}F_{\mu\nu}x^\nu \qquad (\hat{n}\cdot\hat{n} = 1)\,. \tag{2.6}$$

Here, λ^ℓ are the Gell-Mann matrices. We regard it convenient for our purpose to rotate the space coordinate system in such a way that the $\hat{x}^3$-axis is along the QCD-"electric" field $\vec{\mathcal{E}}$, and the $\hat{x}^2$-axis is along the component of the QCD-"magnetic" field $\vec{\mathcal{B}}_\perp$ perpendicular to $\vec{\mathcal{E}}$. Stated otherwise, without loss of generality we take $\vec{\mathcal{E}} = (0, 0, \mathcal{E})$ (with $\mathcal{E} > 0$), and $\vec{\mathcal{B}} = (0, \mathcal{B}_\perp, \mathcal{B}_\|)$ (with $\mathcal{B}_\perp > 0$). Furthermore, it appears convenient for our purposes to change the gauge in such a way that the B^μ field acquires the somewhat simpler form

$$B^\mu = \left(\frac{\hat{n}\cdot\vec{\lambda}}{2}\right)(0, +\mathcal{B}_\perp x^3, +\mathcal{B}_\| x^1, -\mathcal{E}x^0)\,. \tag{2.7}$$

This can be obtained from the form (2.6), after the mentioned rotation of space coordinate system, by the gauge transformation: $G = \exp[\mathrm{i}(\hat{n}\cdot\vec{\lambda})f(x)]$, with $f(x) = g_s(-\mathcal{E}x^3x^0 + \mathcal{B}_\| x^2x^1 + \mathcal{B}_\perp x^3x^1)/4$. Such gauge transformations, since they are along the $(\hat{n};\vec{\lambda})$-direction, do not change the antisymmetric field tensor (2.4).

Path integrals (2.2), (2.3), which determine w through (2.1), lead directly to the expression

$$w = -\frac{1}{\Omega}\mathrm{Re}\{\mathrm{Tr}\ln[(\widehat{\not{P}} - g_s\not{B})^2 - (m-\mathrm{i}\varepsilon)^2] - \mathrm{Tr}\ln[\widehat{\not{P}}^2 - (m-\mathrm{i}\varepsilon)^2]\}\,, \tag{2.8}$$

where Tr denotes tracing over all the relevant degrees of freedom—over configuration space (*e.g.*, integral over $\mathrm{d}^4x\langle x|\cdots|x\rangle$), and over spinor and color degrees of freedom. Using the identity

$$(\widehat{\not{P}} - g_s\not{B})^2 = (\widehat{P} - g_sB)^2 - \frac{g_s}{2}\sigma_{\mu\nu}(\partial^\mu B^\nu - \partial^\nu B^\mu),\ \text{where}\ \sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]\,, \tag{2.9}$$

and the Schwinger integral representation for logarithms in terms of proper time s, we obtain(2) from (2.8)

$$w = \mathrm{Re}\int_0^\infty \frac{\mathrm{d}s}{s}\exp[-\mathrm{i}s(m^2-\mathrm{i}\varepsilon)]\{\mathrm{tr}\langle x|\exp[\mathrm{i}s(\widehat{P} - g_sB(\widehat{X}))^2]\times$$
$$\times\exp[-\mathrm{i}s(g_s/2)(\hat{n}\cdot\vec{\lambda}/2)\sigma_{\mu\nu}F^{\mu\nu}]|x\rangle - \mathrm{tr}\langle x|\exp[\mathrm{i}s\widehat{P}^2]|x\rangle\}\,. \tag{2.10}$$

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(2) We use shorthand notation: $(\widehat{P}^\mu - g_sB^\mu)(\widehat{P}_\mu - g_sB_\mu) \equiv (\widehat{P} - g_sB)^2$.

To obtain this relation, we canceled $\int \mathrm{d}^4x$ with the four-dimensional volume Ω in the denominator, because w was assumed to be x-independent. In retrospect, this assumption will turn out to be correct, a consequence of the fact that the gauge field is taken to be homogeneous, time-independent and extended over a large space. Furthermore, in (2.10) we rewrote $(m - \mathrm{i}\varepsilon)^2$ in the equivalent form $(m^2 - \mathrm{i}\varepsilon)$, where the infinitesimal $\varepsilon = +0$ ensures convergence of the integral.

Due to homogeneity of the B gauge field, we will be able to evaluate expression (2.10) explicitly. The tracing "tr" in this expression is over the spinor (sp) and color (c) degrees of freedom: $\mathrm{tr} = \mathrm{tr_c tr_{sp}}$. Only the term involving $\sigma_{\mu\nu}$ has a nontrivial (4×4) spinor structure. Direct evaluation (Appendix A) yields

$$\mathrm{tr_{sp}} \exp[-\mathrm{i}s(g_s/2)(\hat{n} \cdot \vec{\lambda}/2)\sigma_{\mu\nu}F^{\mu\nu}] = 4\cosh[(\hat{n} \cdot \vec{\lambda}/2)g_s s\tilde{a}] \cos[(\hat{n} \cdot \vec{\lambda}/2)g_s s\tilde{b}]\,, \tag{2.11}$$

where these expressions are to be understood as 3×3 matrices in the quark color space, and the Lorentz-invariant parameters $\tilde{a}$, $\tilde{b}$ are defined in terms of the QCD-"electric" and QCD-"magnetic" field strengths,

$$\tilde{a} = \left[+\vec{\mathcal{E}}^2 - \vec{\mathcal{B}}^2 + \sqrt{(\vec{\mathcal{E}}^2 - \vec{\mathcal{B}}^2)^2 + 4(\vec{\mathcal{E}} \cdot \vec{\mathcal{B}})^2}\right]^{1/2} \Big/ \sqrt{2}\,, \tag{2.12}$$

$$\tilde{b} = \left[-\vec{\mathcal{E}}^2 + \vec{\mathcal{B}}^2 + \sqrt{(\vec{\mathcal{E}}^2 - \vec{\mathcal{B}}^2)^2 + 4(\vec{\mathcal{E}} \cdot \vec{\mathcal{B}})^2}\right]^{1/2} \Big/ \sqrt{2}\,. \tag{2.13}$$

Note that $\tilde{a}\tilde{b} = |\vec{\mathcal{E}} \cdot \vec{\mathcal{B}}|$, and $\tilde{a}^2 - \tilde{b}^2 = \vec{\mathcal{E}}^2 - \vec{\mathcal{B}}^2$. The other matrix element in (2.10) can be evaluated by transforming operator $[\widehat{P} - g_s B(\widehat{X})]^2$ via a unitary transformation into a sum of two harmonic-oscillator Hamiltonian densities:

$$\widehat{U}[\widehat{P} - g_s B(X)]^2 \widehat{U}^\dagger = |\alpha_1| \left[\widehat{P}^0 \widehat{P}^0 - \frac{g_s^2 \tilde{A}^2}{|\alpha_\mathrm{i}|^2} \widehat{X}^0 \widehat{X}^0\right] - |\alpha_2| \left[\widehat{P}^1 \widehat{P}^1 + \frac{g_s^2 \widetilde{B}^2}{|\alpha_2|^2} \widehat{X}^1 \widehat{X}^1\right]\,, \tag{2.14}$$

with

$$\tilde{A} = \tilde{a}(\hat{n} \cdot \vec{\lambda}/2), \qquad \widetilde{B} = \tilde{b}(\hat{n} \cdot \vec{\lambda}/2)\,. \tag{2.15}$$

Unitary operator $\widehat{U}$ is constructed in Appendix B, where also expressions for $|\alpha_j|$ ($j = 1, 2$) are given. Inserting (2.14) and (2.11) into expression (2.10), and using unitary of $\widehat{U}$ and completeness of the four-momentum states $|q\rangle$, we obtain

$$\begin{aligned} w = 4\mathrm{Re} \int_0^\infty \frac{\mathrm{d}s}{s} \exp[-\mathrm{i}s(m^2 - \mathrm{i}\varepsilon)]\mathrm{tr_c}\Big\{&\cosh(\tilde{A} g_s s) \cos(\widetilde{B} g_s s) \times \\ &\times \iint \mathrm{d}^4q\mathrm{d}^4q' \langle x|\widehat{U}^\dagger|q\rangle \langle q^2|q'^2\rangle \langle q^3|q'^3\rangle \times \\ &\times \left\langle q^0 \left| \exp\left[\mathrm{i}s2|\alpha_1| \left(\frac{1}{2}\widehat{P}^0 \widehat{P}^0 - \frac{g_s^2 \tilde{A}^2}{2|\alpha_1|^2} \widehat{X}^0 \widehat{X}^0\right)\right] \right| q'^0 \right\rangle \times \\ &\times \left\langle q^1 \left| \exp\left[-\mathrm{i}s2|\alpha_2| \left(\frac{1}{2}\widehat{P}^1 \widehat{P}^1 + \frac{g_s^2 \widetilde{B}^2}{2|\alpha_2|^2} \widehat{X}^1 \widehat{X}^1\right)\right] \right| q'^1 \right\rangle \langle q'|\widehat{U}|x\rangle - \dots \Big\}\,, \end{aligned} \tag{2.16}$$

where the dots stand for the same expression but with no field ($\tilde{A}$, $\widetilde{B} \mapsto 0$). We note that integrations over q'^2 and q^3 in (2.16) are trivial because $\langle q^j|q'^j\rangle = 2\pi\delta(q^j - q'^j)$. Matrix elements involving $\widehat{U}$ and $\widehat{U}^\dagger$ are calculated in Appendix C, where it is subsequently shown that integrations over q^2, q^3, q'^0 and q'^1 can be explicitly performed leading to

$$(2.17)\quad w = \frac{1}{4\pi^2}\mathrm{Re}\int_0^\infty \frac{\mathrm{d}s}{s}\exp[-\mathrm{i}s(m^2-\mathrm{i}\varepsilon)]\Bigg\{\mathrm{tr_c}4g_s^2\tilde{A}\widetilde{B}\cosh(\tilde{A}g_s s)\cos(\widetilde{B}g_s s)\times \\ \times\int \mathrm{d}q^0\left\langle q^0\left|\exp\left[\mathrm{i}s2|\alpha_1|\left(\frac{1}{2}\widehat{P}^0\widehat{P}^0 - \frac{g_s^2\tilde{A}^2}{2|\alpha_1|^2}\widehat{X}^0\widehat{X}^0\right)\right]\right|q^0\right\rangle\times \\ \times\int \mathrm{d}q^1\left\langle q^1\left|\exp\left[-\mathrm{i}s2|\alpha_2|\left(\frac{1}{2}\widehat{P}^1\widehat{P}^1 + \frac{g_s^2\widetilde{B}^2}{2|\alpha_2|^2}\widehat{X}^1\widehat{X}^1\right)\right]\right|q^1\right\rangle - \dots\Bigg\},$$

where dots denote again the analogous term with zero fields. Expressions in the above parentheses in the exponents are in fact Hamiltonian densities with mass parameter $m = 1$ and with 3×3-matrix frequency parameters $\omega = \mathrm{i}g_s\tilde{A}/|\alpha_1|$ and $\omega = g_s\widetilde{B}/|\alpha_2|$, respectively. Integration over $\mathrm{d}q^0$ ($\mathrm{d}q^1$) amounts to tracing over configuration space for exponents of these oscillators. Since the result of tracing is independent of the chosen basis, it is convenient to choose for the basis the known eigenstates of these oscillators. This leads to a discrete geometric sum of 3×3 (color) matrices, as shown explicitly in Appendix D, and results in

$$(2.18)\quad w = \frac{1}{16\pi^2}\mathrm{Re}\int_{-\mathrm{i}\varepsilon'}^{\infty-\mathrm{i}\varepsilon'}\mathrm{d}z\frac{1}{\mathrm{i}z}\exp[-\mathrm{i}z(m^2-\mathrm{i}\varepsilon)]\times \\ \times\left\{\mathrm{tr_c}\left[\frac{4g_s^2\tilde{A}\widetilde{B}\cosh(\tilde{A}g_s z)\cos(\widetilde{B}g_s z)}{\sinh(\tilde{A}g_s z)\sin(\widetilde{B}g_s z)}\right] - \frac{12}{z^2}\right\}.$$

We point out that the 3×3 color matrices $\tilde{A}$ and $\widetilde{B}$ of (2.15), as well as their inverses, are Hermitean and commute with each other. This is then true also for any real matrix functions of $\tilde{A}$ and $\widetilde{B}$. That is why the order of matrix factors in (2.16)-(2.18) did not matter. Factor $12 = 4\cdot 3$ in the last (zero-field) term in (2.18) arises from tracing over spinor and color degrees of freedom. In the above expression, when compared with (2.17), we moved the integration slightly (infinitesimally) below the positive real axis ($s \mapsto z = s - \mathrm{i}\varepsilon'$), in order to ensure that the geometric sum originating from tracing over $\mathrm{d}q^1$ (Appendix D) converges. At this point, it is convenient to rewrite the above integral as an integral along the entire real axis. Namely, if we denote the integrand as $\mathrm{tr_c}I_\varepsilon(z)$, it is straightforward to show that $I_\varepsilon(-z^*) = (I_{-\varepsilon}(z))^*$. Hermiticity of matrices $\tilde{A}$ and $\widetilde{B}$ is crucial for showing this identity. Therefore, we can replace $\mathrm{Re}I_\varepsilon(z)$ by $[I_\varepsilon(z) + I_{-\varepsilon}(-z^*)]/2$. If we perform in addition the limit $\varepsilon \to 0$, we obtain

$$(2.19)\quad w = \frac{1}{32\pi^2}\int_{-\infty-\mathrm{i}\varepsilon'}^{+\infty-\mathrm{i}\varepsilon'}\frac{\mathrm{d}z}{\mathrm{i}z}\exp[-\mathrm{i}zm^2]\left\{\mathrm{tr_c}\left[\frac{4g_s^2\tilde{A}\widetilde{B}\cosh(\tilde{A}g_s z)\cos(\widetilde{B}g_s z)}{\sinh(\tilde{A}g_s z)\sin(\widetilde{B}g_s z)}\right] - \frac{12}{z^2}\right\}.$$

Taking the limit $\varepsilon \to 0$ in the mass terms $m^2 \pm \mathrm{i}\varepsilon$ appearing in the exponent is legitimate only if the resulting total integral above is convergent. We will see below that this is the case, thus justifying the limit retrospectively. The form (2.19) is almost ready for

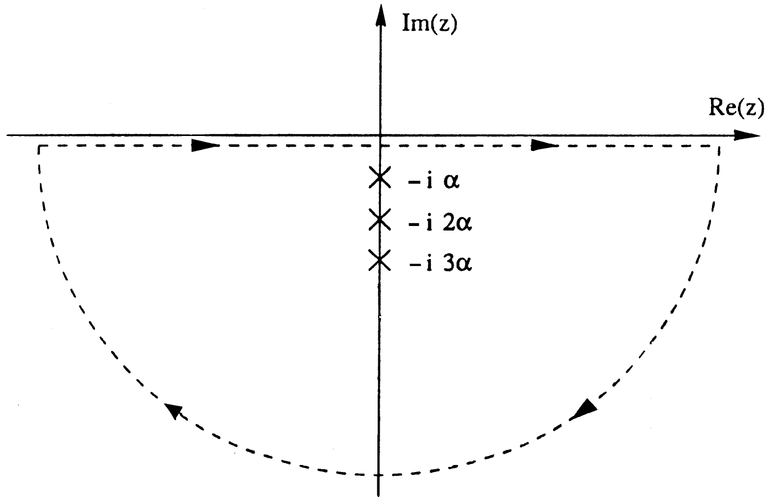

Fig. 1. – Path in the complex plane used for evaluation of integral (2.21). The semicircle has radius $R \to \infty$. Enclosed poles of the integrand are at $z_k = -\mathrm{i}k\alpha$, where $k = 1, 2, \ldots$ and $\alpha = 2\pi/[g_s\tilde{a}|(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}_{jj}|]$ $(j = 1, 2, 3)$.

integration via residue theorem. The tracing over color degrees of freedom ($\mathrm{tr_c}$) can be performed explicitly when unitarily rotating the Hermitean matrix $(\hat{n}\cdot\vec{\lambda})$ into its real diagonal form $(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}$,

$$(\hat{n}\cdot\vec{\lambda}/2) = \mathcal{U}(\hat{n})(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}\mathcal{U}^{\dagger}(\hat{n}), \tag{2.20}$$

leading to the explicit c-number integrand

$$w = \frac{1}{8\pi^2}\int_{-\infty-\mathrm{i}\varepsilon'}^{+\infty-\mathrm{i}\varepsilon'}\frac{\mathrm{d}z}{\mathrm{i}z}\exp[-\mathrm{i}zm^2]\times \times\left\{(g_s^2\tilde{a}\tilde{b})\sum_{j=1}^{3}\frac{[(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}_{jj}/2]^2\cosh[zg_s\tilde{a}(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}_{jj}/2]\cos[zg_s\tilde{b}(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}_{jj}/2]}{\sinh[zg_s\tilde{a}(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}_{jj}/2]\sin[zg_s\tilde{b}(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}_{jj}/2]}-\frac{3}{z^2}\right\}. \tag{2.21}$$

In the above formula, we can replace for convenience $(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}_{jj} \mapsto |(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}_{jj}|$, since the integrand remains unchanged under this replacement. We can close the integration path in (2.21) with an (infinitely) large semicircle in the lower half of the complex plane—we can check that this part of the path contributes zero because $\exp[-\mathrm{i}zm^2]$ exponentially decreases for $\mathrm{Im}(z) < 0$. Then, by the residue theorem the integral is proportional to the sum of residues (cf. fig. 1) at the enclosed poles $z_k = -\mathrm{i}k2\pi/[g_s\tilde{a}|(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}_{jj}|]$ $(k = 1, 2, \ldots)$—the latter poles originate from sinh in the denominator of (2.21). The relevant expression for calculation of the corresponding residues is

$$\lim_{z\to z_k}\frac{(z-z_k)}{\sinh(z\alpha)} = (-1)^k/\alpha\,. \tag{2.22}$$

The resulting sum of residues and hence the final expression for the quark-antiquark production probability density $w \equiv \mathrm{d}P/\mathrm{d}^4x$, is then directly obtained:

$$w = \frac{g_s^2}{4\pi}\frac{\tilde{a}\tilde{b}}{\pi}\sum_{k=1}^{\infty}\sum_{j=1}^{N_c}\frac{1}{k}[(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}]^2 \exp\left[-k\frac{2\pi m^2}{g_s\tilde{a}|(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}_{jj}|}\right]\coth\left[k\frac{\pi\tilde{b}}{\tilde{a}}\right]. \tag{2.23}$$

We denoted the number of colors as $N_c(=3)$. We recall that the positive Lorentz-invariant field strength parameters $\tilde{a}$ and $\tilde{b}$ are given in terms of the QCD-"electric" and QCD-"magnetic" fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ of (2.5) by (2.12), (2.13). Furthermore, $g_s^2/(4\pi) = \alpha_s(\mu)$ is the strong QCD coupling parameter at low energies of the quark mass $\mu \sim m$. Strictly speaking, summation in (2.23) should run not just over the pole index (k) and color degrees of freedom (j), but also over quark flavors (sum over $m = m_u, m_d, m_s, \ldots$). However, if the field ($\propto \tilde{a}$) is not extremely strong, only the lightest flavor u (and possibly d) would contribute due to the $\exp[-cm^2]$ suppression factor.

Finally, we note that the derivation here was performed for the case when $(\hat{n}\cdot\vec{\lambda})$ is an invertible matrix. If it is not (*e.g.*, for $\hat{n}\cdot\vec{\lambda} = \lambda^\ell$, $\ell \neq 8$), then at least one eigenvalue is zero, say $(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}_{33} = 0$. However, expression (2.23) can be applied also in such a case, since the limit $(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}_{33} \to 0$ can be performed continuously there, yielding then contribution zero to w from $j = 3$ terms.

3. – Discussion of the result

The obtained probability density (2.23) shows several interesting properties, some of them general, and others specific to the $SU(3)$ case.

3˙1. *General properties*. – We can discern from QCD probability density (2.23) several general properties which apply also in the $U(1)$ (electromagnetic-EM) and $SU(2)$ cases.

a) If the (QCD)-"electric" field $\vec{\mathcal{E}}$ is negligible ($\tilde{a} \to +0$, $\tilde{b} \to |\vec{\mathcal{B}}|$), the pair production probability density w becomes negligible, too. Therefore, without the "electric" component there is no production probability.

b) If the (QCD)-"magnetic" field $\vec{\mathcal{B}}$ is negligible ($\tilde{a} \to |\vec{\mathcal{E}}|$, $\tilde{b} \to +0$), (2.23) reduces to

$$w = \frac{g_s^2}{4\pi}\frac{|\vec{\mathcal{E}}|^2}{\pi^2}\sum_{k=1}^{\infty}\sum_{j=1}^{N_c}\frac{1}{k^2}[(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}]^2 \exp\left[-k\frac{2\pi m^2}{g_s|\vec{\mathcal{E}}||(\hat{n}\cdot\vec{\lambda})^{\mathrm{d}}_{jj}|}\right]. \tag{3.1}$$

In the $U(1)$ (electric) case $[(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj} \mapsto 1,\ g_s^2 \mapsto e^2,\ N_c \mapsto 1]$, this reduces to the well-known formula of Schwinger—eq. (6.41) of ref. [1], eq. (4-118) of ref. [5]. However, the additional presence of (QCD)-"magnetic" field ([3]) does modify the probability density appreciably. In the general $U(1)$ case (EM), the proper time integral expression (2.18) for w reduces, under the mentioned QCD $\mapsto$ EM replacements, to $w = 2\mathrm{Im}(\delta\mathcal{L})$, where $\delta\mathcal{L}$ is the Euler-Heisenberg effective Lagrangian density [6]. Stated otherwise, the pair production by a gauge field is described by the absorptive part of the quantum-corrected

([3]) The latter field may be created by moving quarks.

gauge field Lagrangian density $\delta\mathcal{L}$. The electromagnetic $\delta\mathcal{L}$ was also derived in an integral form involving the rotated proper time ($z \mapsto -\mathrm{i}z$) by Schwinger [1] (eq. (3.44) there).

c) In fact, $\delta\mathcal{L}$ for the (covariantly homogeneous) QCD field case can be read off from (2.18)

$$\delta\mathcal{L} = \frac{1}{8\pi^2}\int_{-\mathrm{i}\varepsilon'}^{\infty-\mathrm{i}\varepsilon'} \frac{\mathrm{d}z}{z} e^{-\mathrm{i}z(m^2-\mathrm{i}\varepsilon)} \left\{ \mathrm{tr_c}\left[\frac{g_s^2 \tilde{A}\widetilde{B}\cosh(\tilde{A}g_s z)\cos(\widetilde{B}g_s z)}{\sinh(\tilde{A}g_s z)\sin(\widetilde{g}_s z)}\right] - \frac{3}{z^2}\right\}, \tag{3.2}$$

with $\mathcal{A}$ and $\mathcal{B}$ given by (2.15) and (2.12), (2.13).

d) Straightforward probability calculation allows one to obtain in general from the pair creation probability density $w(x) \equiv \mathrm{d}P/\mathrm{d}^4x$ various relevant probabilities. For example, the probability $P_1[\mathrm{d}^4x_i]$ that one pair is created in an infinitesimally small space-time volume d^4x_i (and in the rest of the space-time nothing happens) is in general

$$\begin{aligned} P_1[\mathrm{d}^4x_i] &= \mathrm{d}^4x_i w(x_i) \prod_{n(\neq i)} [1 - \mathrm{d}^4x_n w(x_n)] = \\ &= \mathrm{d}^4x_i w(x_i) \exp\left[-\sum \mathrm{d}^4x_n w(x_n)\right] = \mathrm{d}^4x_i w(x_i) \exp\left[-\int \mathrm{d}^4x w(x)\right]. \end{aligned} \tag{3.3}$$

Integration in the exponent is over the space-time region Ω where the field (and w) is nonzero. From here we can obtain in general the probability $P_n[\Delta^4x]$ that exactly n (quark-antiquark) pairs are created in a finite space-time volume Δ^4x (and nothing in the rest)

$$P_n[\Delta^4x] = \frac{1}{n!}\left[\int_{\Delta^4x} w(x)\right]^n \exp\left[-\int \mathrm{d}^4x w(x)\right] \quad (n = 0, 1, 2, \ldots), \tag{3.4}$$

where the first integration extends over the finite space-time volume Δ^4x, and the second integration in the exponent over the (large) space-time volume Ω where the field is nonzero ($\Delta^4x \subseteq \Omega$). In our specific case, we took the (covariantly) homogeneous field in a large (but in principle finite) space-time volume Ω extending far in all four directions, and therefore obtained an x-independent w (2.23). Therefore, in such a case probability (3.4) reads

$$P_n(\Delta^4x) = \frac{1}{n!}(\Delta^4x)\exp[-\Omega w] \qquad (n = 0, 1, 2, \ldots). \tag{3.5}$$

Equations (3.4) and (3.5) represent Poisson distribution, and it is manifest that $\sum_{n=0}^{\infty} P_n(\Omega) = 1$, as it should be.

e) If we had a QCD theory without quarks, but with charged and colored spinless bosons, then derivation of the production probability density for boson-antiboson pairs would be very similar to that in the previous section. The difference is in the structure of the Lagrangian density $\mathcal{L} = \phi^\dagger[(\widehat{P} - g_s B)^2 - m^2]\phi$ and in the statistics (bosonic instead of Dirac). This leads in (2.8) to the replacement $(\widehat{\not{P}} - g_s \not{B})^2 \mapsto (\widehat{P} - g_s B)^2$ and to an additional overall factor (-2) there. There is no tracing $\mathrm{tr_{sp}}$ over spinor degrees of freedom now. Almost all steps of derivation are identical with the fermionic case, resulting instead of (2.19) in

$$(3.6)\quad w_{\rm sc} = (-2)\frac{1}{32\pi^2}\int_{-\infty-{\rm i}\varepsilon'}^{+\infty-{\rm i}\varepsilon'} \frac{{\rm d}z}{{\rm i}z}\exp[-{\rm i}zm^2]\left\{{\rm tr_c}\left[\frac{g_s^2\tilde{A}\widetilde{B}}{\sinh(\tilde{A}g_s z)\sin(\widetilde{B}g_s z)}\right] - \frac{3}{z^2}\right\}.$$

Performing integration over the complex plane as in the fermionic case gives us the final result for the charged colored scalar case

$$(3.7)\quad w_{\rm sc} = \frac{g_s^2}{4\pi}\frac{\tilde{a}\tilde{b}}{2\pi}\sum_{k=1}^{\infty}\sum_{j=1}^{N_{\rm c}}(-1)^{k+1}\frac{1}{k}[(\hat{n}\cdot\vec{\lambda}/2)^{\rm d}_{jj}]^2\exp\left[-k\frac{2\pi m^2}{g_s\tilde{a}|(\hat{n}\cdot\vec{\lambda})^{\rm d}_{jj}|}\right]/\sinh[k\pi\tilde{b}/\tilde{a}],$$

and in the pure QCD-"electric" case

$$(3.8)\quad w_{\rm sc}(\mathcal{B}=0) = \frac{g_s^2}{4\pi}\frac{\vec{\mathcal{E}}^2}{2\pi^2}\sum_{k=1}^{\infty}\sum_{j=1}^{N_{\rm c}}(-1)^{k+1}\frac{1}{k^2}[(\hat{n}\cdot\vec{\lambda}/2)^{\rm d}_{jj}]^2\exp\left[-k\frac{2\pi m^2}{g_s|\vec{\mathcal{E}}||(\hat{n}\cdot\vec{\lambda})^{\rm d}_{jj}|}\right].$$

3˙2. *Properties specific to the $SU(3)$ case.* – a) Explicit form of the eight Gell-Mann matrices λ^a implies that their eigenvalues are $+1$, -1 and 0 for $a = 1, 2, \ldots, 7$, but for λ^8 the eigenvalues are $1/\sqrt{3}$, $1/\sqrt{3}$ and $-2\sqrt{3}$. The latter property tells us that the value of QCD probability density w (2.23) depends on the specific direction of the QCD field's 8-component real unit vector $\hat{n}$ of (2.6), (2.7), unless $\hat{n}^8 = 0$. This contrasts with the $SU(2)$ case, where $(\hat{n}\cdot\vec{\lambda}) \mapsto (\hat{n}\cdot\vec{\tau})$, and the latter general Pauli matrix has eigenvalues $+1$ and -1 for *any* choice of the $SU(2)$ real unit vector $\hat{n} = (n^1, n^2, n^3)$.

b) One may ask for physical interpretation of the 8-component real unit vector $\hat{n}$ appearing in (equivalent) expressions (2.6), (2.7) for the covariantly homogeneous QCD field. It is probably hard to answer this equation satisfactorily in the nonperturbative context of the QCD flux tube. If we demand that the two end-point quarks of the QCD flux tube form a color singlet, then the (perturbative) picture of one-gluon-exchange between the quark and antiquark would suggest that $(\hat{n}\cdot\vec{\lambda})$ should be diagonal, *i.e.* the one exchanged gluon should mediate between the quark of a given color (say: red) and the antiquark of the same (anti)color (say: anti-red). This would imply that $(\hat{n}\cdot\vec{\lambda})$ would be a linear combination of λ^3 and λ^8 only. This picture was discussed briefly in ref. [3]. However, since the picture is basically perturbative, it is unclear to what degree it can be applied to a strong QCD flux tube field, the latter representing a collective effect of infinitely many gluons with individual energies (almost) zero. Furthermore, there exist relevant processes, *e.g.*, in deep inelastic scattering of leptons on baryons, where the end-point quarks of momentarily created QCD flux tubes may be color-nonsinglets. The subsequent hadronization processes ensure that the final physical particles are colorless.

4. – Conclusions

We derived the formula for the pair production probability density $w \equiv {\rm d}P/{\rm d}^4x$ in a space with a general covariantly homogeneous QCD field. Calculation was performed in the limit when the field stretches over large distances in all space directions, and interactions between the produced particles were ignored (external field approximation).

We believe that the formula provides a reasonable approximation for quark-antiquark productions in so-called QCD flux tubes whose end-point in general move fast. Therefore, fields in such flux tubes are made up of QCD-"electric" and QCD-"magnetic" fields. Such regions of strong field may appear in deep inelastic scattering processes. Confinement does not seem to require in general the color-diagonal structure of such strong fields.

Appendix A.

Evaluation of $\sigma_{\mu\nu}$-term

In this Appendix we derive formula (2.11). Explicit evaluation gives

$$(\sigma_{\mu\nu}F^{\mu\nu})^2 = 4(\mathcal{E}\sigma_{03} + \mathcal{B}_\perp\sigma_{13} + \mathcal{B}_\parallel\sigma_{21})^2 = 4[(-\vec{\mathcal{E}}^2 + \vec{\mathcal{B}}^2) - 2\mathrm{i}\gamma_5(\vec{\mathcal{E}}\cdot\vec{\mathcal{B}})]\,. \tag{A.1}$$

It is convenient to use here the following notation for the relevant 3×3 color matrices:

$$\mathcal{F} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}(\hat{n}\cdot\vec{\lambda}/2)^2 = \frac{1}{2}(\vec{\mathcal{B}}^2 - \vec{\mathcal{E}}^2)(\hat{n}\cdot\vec{\lambda}/2)^2\,, \tag{A.2}$$

$$\mathcal{G} = -\frac{1}{4}F^{\mu\nu}\widetilde{F}_{\mu\nu}(\hat{n}\cdot\vec{\lambda}/2)^2 = (\vec{\mathcal{E}}\cdot\vec{\mathcal{B}})(\hat{n}\cdot\vec{\lambda}/2)^2\,. \tag{A.3}$$

Expanding the exponent on the left of (2.11), and using identity (A.1) and tracelessness of $\sigma_{\mu\nu}$, γ_5 and of $\gamma_5\sigma_{\mu\nu}$, then leads to

$$\begin{aligned} &\mathrm{tr_{sp}}\exp[-\mathrm{i}s(g_s/2)(\hat{n}\cdot\vec{\lambda}/2)\sigma_{\mu\nu}F^{\mu\nu}] = \\ &= 4\Big\{1 - \frac{g_s^2 2^1 s^2}{2!}\mathcal{F} + \frac{g_s^4 2^2 s^4}{4!}(\mathcal{F}^2 - \mathcal{G}^2) + \ldots + (-1)^n\frac{g_s^{2n}2^n s^{2n}}{(2n)!}\times \\ &\times\Big[\mathcal{F}^n - \binom{n}{2}\mathcal{F}^{n-2}\mathcal{G}^2 + \ldots + (-1)^k\binom{n}{2k}\mathcal{F}^{n-2k}\mathcal{G}^{2k} + \ldots\Big] + \ldots\Big\} \end{aligned} \tag{A.4}$$

$$\begin{aligned} &= 2\Big\{(1{+}1) - \frac{g_s^2 2^1 s^2}{2!}[(\mathcal{F}{-}\mathrm{i}\mathcal{G}) + (\mathcal{F}{+}\mathrm{i}\mathcal{G})] + \frac{g_s^4 2^2 s^4}{4!}[(\mathcal{F}{-}\mathrm{i}\mathcal{G})^2 + (\mathcal{F}{+}\mathrm{i}\mathcal{G})^2] + \ldots \\ &+(-1)^n\frac{g_s^{2n}2^n s^{2n}}{(2n)!}[(\mathcal{F}-\mathrm{i}\mathcal{G})^n + (\mathcal{F}+\mathrm{i}\mathcal{G})^n] + \ldots\Big\} \end{aligned} \tag{A.5}$$

$$= 2\Big\{\cos[g_s s\sqrt{2}(\mathcal{F}-\mathrm{i}\mathcal{G})^{1/2}] + \cos[g_s s\sqrt{2}(\mathcal{F}+\mathrm{i}\mathcal{G})^{1/2}]\Big\}\,. \tag{A.6}$$

It is straightforward to see that $2(\mathcal{F}\pm\mathrm{i}\mathcal{G}) = (\widetilde{B}\pm\mathrm{i}\tilde{A})^2$, where the 3×3 color matrices $\tilde{A}$ and $\widetilde{B}$ are defined through (2.15) and (2.12), (2.13). Therefore, we can rewrite (A.6) as

$$\mathrm{tr_{sp}}\exp[-\mathrm{i}s(g_s/2)(\hat{n}\cdot\vec{\lambda}/2)\sigma_{\mu\nu}F^{\mu\nu}] = 2\{\cos[(\widetilde{B}-\mathrm{i}\tilde{A})g_s s] + \cos[(\widetilde{B}+\mathrm{i}\tilde{A})g_s s]\} \tag{A.7}$$

$$= 4\cosh(\tilde{A}g_s s)\cos(\widetilde{B}g_s s)\,. \tag{A.8}$$

This is identity (2.11), with notations (2.15) used.

Appendix B.

Unitary transformation leading to harmonic oscillators

We want to find the unitary operator $\widehat{U}$ which "diagonalizes" the scalar density for the general covariantly homogeneous QCD field (2.7),

$$[\widehat{P} - g_s B]^2 = (\widehat{P}^0)^2 - (\widehat{P}^1 - G_s \mathcal{B}_\perp \widehat{X}^3)^2 - (\widehat{P}^2 - G_s \mathcal{B}_\parallel \widehat{X}^1)^2 - (\widehat{P}^3 + G_s \mathcal{E} \widehat{X}^0)^2 , \tag{B.1}$$

where

$$G_s = g_s(\hat{n} \cdot \vec{\lambda}/2) , \tag{B.2}$$

into the sum of two harmonic-oscillator densities as written in (2.14). This approach can be regarded as a generalization of the approach by Itzykson and Zuber [5]—the latter authors used a relatively simple unitary "diagonalizing" operator for the case of homogeneous electromagnetic field. However, in our case the unitary operator $\widehat{U}$ will be more complicated, basically because we include in addition to the (QCD)-"electric" field also the (QCD)-"magnetic" field in an arbitrary direction. Forms (B.1) and (2.14), and the commutation relations $[\widehat{X}^\mu, \widehat{P}^\nu] = -\mathrm{i}g^{\mu\nu}$, suggest that operator $\widehat{U}$, which incidentally is a 3×3 matrix in the color space, must be a function of operators X^μ and $P^\nu = \mathrm{i}\partial^\nu$. In fact, using the mentioned commutation relations and the operator identity([4])

$$\exp[\widehat{B}]\hat{A}\exp[-\widehat{B}] = \hat{A} + \frac{1}{1!}[\widehat{B}, \widehat{A}] + \frac{1}{2!}[\widehat{B}, [\widehat{B}, \hat{A}]] + \ldots \frac{1}{n!}[\widehat{B}, [\widehat{B}, \ldots [\widehat{B}, \hat{A}] \ldots]] + \ldots , \tag{B.3}$$

we can derive the following useful formulas:

$$\exp[\mathrm{i}\alpha \widehat{X}^\mu \widehat{P}^\nu] \begin{bmatrix} \widehat{X}^\delta \\ \widehat{P}^\delta \end{bmatrix} \exp[-\mathrm{i}\alpha \widehat{X}^\mu \widehat{P}^\nu] = \begin{bmatrix} \widehat{X}^\delta \\ \widehat{P}^\delta \end{bmatrix} + \alpha \begin{bmatrix} -g^{\nu\delta} \widehat{X}^\mu \\ +g^{\mu\delta} \widehat{P}^\nu \end{bmatrix} \qquad (\mu \neq \nu) ; \tag{B.4}$$

$$\exp[\mathrm{i}\alpha \widehat{X}^\mu \widehat{X}^\nu](\widehat{P}^\delta) \exp[-\mathrm{i}\alpha \widehat{X}^\mu \widehat{X}^\nu] = \widehat{P}^\delta + \alpha g^{\nu\delta} \widehat{X}^\mu + \alpha g^{\mu\delta} \widehat{X}^\nu \quad (\mu \neq \nu) ; \tag{B.5}$$

$$\exp[\mathrm{i}\alpha \widehat{P}^\mu \widehat{P}^\nu)(\widehat{X}^\delta) \exp[-\mathrm{i}\alpha \widehat{P}^\mu \widehat{P}^\nu] = \widehat{X}^\delta - \alpha g^{\nu\delta} \widehat{P}^\mu - \alpha g^{\mu\delta} \widehat{P}^\nu \quad (\mu \neq \nu) . \tag{B.6}$$

We can now make the following ansatz for the unitary operator $\widehat{U}$:

$$\widehat{U} = \exp[\mathrm{i}\xi_5 \widehat{X}^1 \widehat{P}^0] \exp[\mathrm{i}\xi_4 \widehat{X}^0 \widehat{P}^1] \exp[\mathrm{i}\xi_3 \widehat{X}^1 \widehat{X}^3] \exp[\mathrm{i}\xi_2 \widehat{P}^1 \widehat{P}^2] \exp[\mathrm{i}\xi_1 \widehat{P}^0 \widehat{P}^3] . \tag{B.7}$$

Using formulas (B.5) and (B.6) and unitarity of $\widehat{U}$, we can show that it is possible to choose the first three color-matrix parameters ξ_j $(j = 1, 2, 3)$ in such a way that the following three requirements are fulfilled: $\widehat{P}^k$ $(k = 2, 3)$ and $\widehat{X}^3$ operators in (B.1) vanish,

$$\widehat{U}_0[\widehat{P} - g_s B(\widehat{X})]^2 \widehat{U}_0^\dagger = (\widehat{P}^0)^2 - \left(\widehat{P}^1 - \frac{\mathcal{B}_\perp}{\mathcal{E}} \widehat{P}^0\right)^2 - G_s^2 \mathcal{E}^2 (\widehat{X}^0)^2 - G_s^2 \mathcal{B}_\parallel^2 (\widehat{X}^1)^2 , \tag{B.8}$$

([4]) This identity can be proved, for example, by introducing operator $\widehat{F}(\alpha) = \exp[\alpha\widehat{B}]\hat{A}\exp[-\alpha\widehat{B}]$; it is straightforward to show that $\mathrm{d}^n\widehat{F}(\alpha)/\mathrm{d}\alpha^n = [\widehat{B}, [\widehat{B}, \ldots [\widehat{B}, \widehat{F}] \ldots]]$, where the latter expression involves n commutators; Taylor expansion of $\widehat{F}(1)$ around the point $\alpha = 0$ then leads to (B.3).

where

$$\widehat{U}_0 = \exp[-\mathrm{i}G_s\mathcal{B}_\perp \widehat{X}^1\widehat{X}^3]\exp[\mathrm{i}G_s^{-1}\mathcal{B}_\parallel^{-1}\widehat{P}^1\widehat{P}^2]\exp[\mathrm{i}G_s^{-1}\mathcal{E}^{-1}\widehat{P}^0\widehat{P}^3]\,. \tag{B.9}$$

Here we implicitly assumed that matrix G_s (B.2) is invertible. However, the final derived formulas for pair production probability densities will be applicable also in the case when G_s is not invertible—cf. discussion at the end of sect. **2**. At the second stage, we can use formulas (B.4) (and unitarity of $\widehat{U}$) to determine ξ_4 and ξ_5 in (B.7) in such way that the mixing term $\widehat{P}^0\widehat{P}^1$ in (B.8) is eliminated and at the same time no mixing term $\widehat{X}^0\widehat{X}^1$ is generated (*i.e.* two requirements). The algebra involved is straightforward, but somewhat lengthy. The final result is

$$\widehat{U}[\widehat{P} - g_sB(\widehat{X})]^2\widehat{U}^\dagger = \left[|\alpha_1|(\widehat{P}^0)^2 - \frac{G_s^2\tilde{a}^2}{|\alpha_1|}(\widehat{X}^0)^2\right] - \left[|\alpha_2|(\widehat{P}^1)^2 + \frac{G_s^2\tilde{b}^2}{|\alpha_2|}(\widehat{X}^1)^2\right], \tag{B.10}$$

where the 3×3 color matrix G_s is defined in (B.2), the positive real parameters $\tilde{a}$ and $\tilde{b}$ are given in terms of the fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ in (2.12), (2.13), and

$$\widehat{U} = \exp[\mathrm{i}\xi_5\widehat{X}^1\widehat{P}^0]\exp[\mathrm{i}\xi\widehat{X}^0\widehat{P}^1]\widehat{U}_0\,, \tag{B.11}$$

with $\mathcal{U}_0$ given in (B.9), and the 3×3 matrix parameters ξ_4 and ξ_5 are proportional to 3×3 unit color matrix I and are given by

$$\xi_4 = I\frac{1}{2}(\zeta - \eta), \quad \xi_5 = -I\frac{1}{\eta}\,, \tag{B.12}$$

where

$$\zeta = \frac{\mathcal{E}}{\mathcal{B}_\perp\mathcal{B}_\parallel^2}(\mathcal{E}^2 + \mathcal{B}_\parallel^2 - \mathcal{B}_\perp^2), \quad \eta = \frac{\mathcal{E}}{\mathcal{B}_\perp\mathcal{B}_\parallel^2}\sqrt{(\vec{\mathcal{E}}^2 - \vec{\mathcal{B}}^2)^2 + 4(\vec{\mathcal{E}}\cdot\vec{\mathcal{B}})^2}\,. \tag{B.13}$$

The positive constants $|\alpha_j|$ are given by

$$|\alpha_1| = \frac{\tilde{a}^2}{[(\zeta-\eta)^2\mathcal{B}_\parallel^2/4 + \mathcal{E}^2]}, \quad |\alpha_2| = \frac{\tilde{b}^2\eta^2}{[(\zeta+\eta)^2\mathcal{B}_\parallel^2/4 + \mathcal{E}^2]}\,. \tag{B.14}$$

Formula (B.10) is in fact relation (2.14) in the main text which is thus proved. Operator $\widehat{U}$ as determined in (B.11) and (B.9) is really unitary, because Gell-Mann matrices λ^a and hence G_s of (B.2) are Hermitean matrices. Expressions for parameters ξ_j ($j = 1,\ldots,5$) of the unitary transformation $\widehat{U}$, as well as constants $|\alpha_k|$ ($k = 1,2$) in (B.10), are not Lorentz-invariant quantities. However, they do not appear in the final physical results such as the pair production probability density w.

Appendix C.

Configuration space integrals for probability density

In this Appendix we show that in expression (2.16) integrations over q'^2, q'^3, q^2, q^3 and over q'^0, q'^1 can be explicitly performed, leading to the simplified expression (2.17) involving only traces of exponents of two one-dimensional harmonic oscillators. First we calculate matrix element $\langle x|\widehat{U}^\dagger|q\rangle$. We use expressions (B.11), (B.9) and (B.2) which determine $\widehat{U}$ (and $\widehat{U}^\dagger$), and insert several times completeness relations for the space-time eigenstates $|x_{(j)}\rangle$ and four-momentum eigenstates $|q_{(j)}\rangle$:

$$\text{(C.1)}\quad \langle x|\widehat{U}^\dagger|q\rangle = \int \mathrm{d}^4q_{(1)}\mathrm{d}^4q_{(2)}\mathrm{d}^4x_{(1)}\mathrm{d}^4x_{(2)}\mathrm{d}^4x_{(3)} \times \\ \times\langle x|\exp[-\mathrm{i}G_s^{-1}\mathcal{E}^{-1}\widehat{P}^0\widehat{P}^3]\exp[-\mathrm{i}G_s^{-1}\mathcal{B}_\parallel^{-1}\widehat{P}^1\widehat{P}^2]|q_{(1)}\rangle\langle q_{(1)}|x_{(1)}\rangle \times \\ \times\langle x_{(1)}|\exp[\mathrm{i}G_s\mathcal{B}_\perp\widehat{X}^1\widehat{X}^3]|x_{(2)}\rangle\langle x_{(2)}|\exp[-\mathrm{i}\xi_4\widehat{X}^0\widehat{P}^1]|q_{(2)}\rangle \times \\ \times\langle q_{(2)}|x_{(3)}\rangle\langle x_{(3)}|\exp[-\mathrm{i}\xi_5\widehat{X}^1\widehat{P}^0]|q\rangle\,.$$

Matrix elements above involving exponential operators include eigenstates of these operators as bra and/or kets. This leads to greatly simplified expressions involving only matrix elements of the type $\langle x_{(j)}|q_{(i)}\rangle = \exp[-\mathrm{i}q_{(i)}\cdot x_{(j)}]/(2\pi)^2$ and c-number exponents. Integrations over $\mathrm{d}^4x_{(2)}$ and $\mathrm{d}x^k_{(3)}$, $\mathrm{d}q^k_{(2)}$ $(k = 0,2,3)$ can then be performed immediately. Integration over $\mathrm{d}x^1_{(3)}$ can then be carried out, leading to a simple c-number delta-function which makes subsequent integration over $\mathrm{d}q^1_{(2)}$ trivial. Out of the remaining integrations over $\mathrm{d}^4q_{(1)}$ and $\mathrm{d}^4x_{(1)}$, integrations over $\mathrm{d}x^0_{(1)}$ and $\mathrm{d}x^2_{(1)}$ lead to c-number delta-functions, making subsequent integrations over $\mathrm{d}q^0_{(1)}$ and $\mathrm{d}q^2_{(1)}$ trivial. The remaining four one-dimensional integrals have a 3×3 color-matrix structure (we rename for simplicity: $x^1_{(1)}\mapsto y^1$, $x^3_{(1)}\mapsto y^3$, $q^1_{(1)}\mapsto p^1$, $q^3_{(1)}\mapsto p^3$)

$$\text{(C.2)}\quad \langle x|\widehat{U}^\dagger|q\rangle = \frac{1}{(2\pi)^4}\int\!\!\int \mathrm{d}y^1\mathrm{d}y^3\exp[\mathrm{i}[G_s\mathcal{B}_\perp y^1y^3 + q^3y^3 + (q^1-\xi_5q^0)y^1]] \times \\ \times\exp[-\mathrm{i}[q^0(1-\xi_4\xi_5)+q^1\xi_4]x^0 + \mathrm{i}q^2x^2] \times \\ \times\int \mathrm{d}p^1\exp[-\mathrm{i}p^1[y^1 - x^1 + G_s^{-1}\mathcal{B}_\parallel^{-1}q^2]] \times \\ \times\int \mathrm{d}p^3\exp[-\mathrm{i}p^3[y^3 - x^3 + G_s^{-1}\mathcal{E}^{-1}(q^0(1-\xi_4\xi_5)+q^1\xi_4)]]\,.$$

Integrations over $\mathrm{d}p^1$ and $\mathrm{d}p^3$ result in delta-functions $(2\pi)\delta(\mathcal{A})$, with $\mathcal{A} = [y^1 - x^1 + G_s^{-1}\mathcal{B}_\parallel^{-1}q^2]$ and $\mathcal{A} = \{y^3 - x^3 + G_s^{-1}\mathcal{E}^{-1}[q^0(1-\xi_4\xi_5)+q^1\xi_4]\}$, respectively([5]). These delta-functions have a 3×3 color-matrix structure and should be understood in the following way:

$$\text{(C.3)}\qquad \delta(\mathcal{A}) = \mathcal{U}(\hat{n})\begin{bmatrix}\delta(\mathcal{A}^{\mathrm{d}}_{11}) & 0 & 0\\ 0 & \delta(\mathcal{A}^{\mathrm{d}}_{22}) & 0\\ 0 & 0 & \delta(\mathcal{A}^{\mathrm{d}}_{33})\end{bmatrix}\mathcal{U}^\dagger(\hat{n})\,,$$

where the 3×3 unitary matrix $\mathcal{U}(\hat{n})$ diagonalizes $(\hat{n}\cdot\vec{\lambda})$ according to (2.20), and hence diagonalizes G_s of (B.2) and the above matrices $\mathcal{A}$. Now integrations $\mathrm{d}y^1\,\mathrm{d}y^3$ can be

([5]) The 3×3 identity matrix I is implicitly understood to appear at those terms which have no color-matrix structure.

trivially performed in (C.2), leading to the final result for the matrix element

$$\langle x|\widehat{U}^{\dagger}|q\rangle = \frac{1}{(2\pi)^2}\exp[\mathrm{i}[q^2\chi_{(2)} + q^3\chi_{(3)} + \chi_{(0)}]]\,, \tag{C.4}$$

where $\chi_{(j)}$ are 3×3 matrices (note: I is the 3×3 unit matrix):

$$\chi_{(2)} = I\left(x^2 - \frac{\mathcal{B}_\perp}{\mathcal{B}_\parallel}x^3\right) + G_s^{-1}\frac{\mathcal{B}_\perp}{\mathcal{B}_\parallel\mathcal{E}}\left[q^0\left(1 - |\xi_4\xi_5 + \frac{\mathcal{E}}{\mathcal{B}_\perp}\xi_5\right) + q^1\left(\xi_4 - \frac{\mathcal{E}}{\mathcal{B}_\perp}\right)\right]\,; \tag{C.5}$$

$$\chi_{(3)} = Ix^3 - G_s^{-1}\frac{1}{\mathcal{E}}[q^0(1-\xi_4\xi_5) + q^1\xi_4]\,; \tag{C.6}$$

$$\chi_{(0)} = -I\left(x^0 + \frac{\mathcal{B}_\perp}{\mathcal{E}}x^1\right)[q^0(1-\xi_4\xi_5) + q^1\xi_4] + Ix^1(-q^0\xi_5 + q^1) + G_s\mathcal{B}_\perp x^1x^3\,. \tag{C.7}$$

Note that numbers ξ_4 and ξ_5 are explicitly given in Appendix B (B.12), (B.13) in terms of $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ fields. Similarly, the other matrix element $\langle q'|\widehat{U}|x\rangle$ in expression (2.16) is

$$\langle q'|\widehat{U}|x\rangle = (\langle x|\widehat{U}^{\dagger}|q'\rangle)^* = \frac{1}{(2\pi)^2}\exp[-\mathrm{i}[q'^2\chi'_{(2)} + q'^3\chi'_{(3)} + \chi'_{(0)}]]\,, \tag{C.8}$$

where expressions for matrices $\chi'_{(j)}$ are the same as those for $\chi_{(j)}$ (C.5)-(C.7), but with replacements $q^0 \mapsto q'^0$ and $q^1 \mapsto q'^1$.

We insert the resulting matrix elements (C.4) and (C.8) into expression (2.16) for the probability density w, and take in addition into account $\langle q^j|q'^j\rangle = 2\pi\delta(q^j - q'^j)$ ($j = 2,3$), which eliminates integrations over $\mathrm{d}q'^2\mathrm{d}q'^3$. Subsequent integrations over $\mathrm{d}q^2\,\mathrm{d}q^3$ lead to matrix-delta-functions $(2\pi)\delta(\chi_{(2)} - \chi'_{(2)})$ and $(2\pi)\delta(\chi_{(3)} - \chi'_{(3)})$, resulting in the expression for w:

$$\begin{aligned} w = \frac{4}{(2\pi)^2}\mathrm{Re}\int_0^\infty \frac{\mathrm{d}s}{s}e^{-\mathrm{i}s(m^2-\mathrm{i}\varepsilon)}\mathrm{tr_c}\Big\{&\cosh(\tilde{A}g_s s)\cos(\widetilde{B}g_s s)\times\\ &\times\iint \mathrm{d}q^0\mathrm{d}q'^0\iint\mathrm{d}q^1\mathrm{d}q'^1\times\\ &\times\left\langle q^0\left|\exp\left[\mathrm{i}s|\alpha_1|\left(\widehat{P}^0\widehat{P}^0 - \frac{g_s^2\tilde{A}^2}{|\alpha_1|^2}\widehat{X}^0\widehat{X}^0\right)\right]\right|q'^0\right\rangle\times\\ &\times\left\langle q^1\left|\exp\left[-\mathrm{i}s|\alpha_2|\left(\widehat{P}^1\widehat{P}^1 + \frac{g_s^2\widetilde{B}^2}{|\alpha_2|^2}\widehat{X}^1\widehat{X}^1\right)\right]\right|q'^1\right\rangle\times\\ &\times\exp[\mathrm{i}(\chi_{(0)} - \chi'_{(0)})]\delta(\chi_{(2)} - \chi'_{(2)})\delta(\chi_{(3)} - \chi'_{(3)}) - \ldots\Big\}\,. \end{aligned} \tag{C.9}$$

The 3×3 matrices $\chi_{(j)} - \chi'_{(j)}$ ($j = 2,3$) are linear combinations of $z_1 \equiv q'^0 - q^0$ and $z_2 \equiv q'^1 - q^1$

$$\chi_{(2)} - \chi'_{(2)} = \mathcal{C}_{11}z_1 + \mathcal{C}_{12}z_2, \quad \chi_{(3)} - \chi'_{(3)} = \mathcal{C}_{21}z_1 + \mathcal{C}_{22}z_2\,, \tag{C.10}$$

where $\mathcal{C}_{ij} = (\hat{n}\cdot\vec{\lambda}/2)^{-1}C_{ij}$ and C_{ij} are real numbers determined by (C.5), (C.6) and the analogous formulas for $\chi'_{(j)}$'s (cf. also definition (B.2) for G_s matrix),

$$C = \frac{\mathcal{B}_\perp}{g_s\mathcal{E}\mathcal{B}_\parallel}\begin{bmatrix} -(1-\xi_4\xi_5+\mathcal{E}\xi_5/\mathcal{B}_\perp) & -(\xi_4-\mathcal{E}/\mathcal{B}_\perp) \\ (1-\xi_4\xi_5)\mathcal{B}_\parallel/\mathcal{B}_\perp & \xi_4\mathcal{B}_\parallel/\mathcal{B}_\perp \end{bmatrix}. \tag{C.11}$$

Now we use formula (C.3) for the resulting matrix delta-functions $\delta[(\hat{n}\cdot\vec{\lambda}/2)^{-1}(C_{k1}z_1 + C_{k2}z_2)]$ (k=1, 2), as well as the property $\delta(\alpha z)$=$\delta(z)/|\alpha|$ for the c-number delta-functions. This allows us to rewrite $\delta(\chi_{(2)}-\chi'_{(2)})\delta(\chi_{(3)}-\chi'_{(3)})$ in (C.9) as $(\hat{n}\cdot\vec{\lambda}/2)^2\delta(C_{1j}z_j)\delta(C_{2j}z_j)$. When we replace integration over $\mathrm{d}q^0\mathrm{d}q^1\mathrm{d}q'^0\mathrm{d}q'^1$ in (C.9) equivalently by integration over $\mathrm{d}q^0\mathrm{d}q^1\mathrm{d}z_1\mathrm{d}z_2$, we can easily integrate out (over $\mathrm{d}z_1\mathrm{d}z_2$) the mentioned c-number delta-functions $\delta(C_{1j}z_j)\delta(C_{2j}z_j)$. These two delta-functions ensure $q'^0 = q^0$ and $q'^1 = q^1$ and result in an additional factor $|\mathrm{Det}(C)|$ in the denominator. The resulting expression is

$$\begin{aligned} w = {} & \frac{1}{\pi^2}\mathrm{Re}\int_0^\infty \frac{\mathrm{d}s}{s}e^{-\mathrm{i}s(m^2-\mathrm{i}\varepsilon)}\mathrm{tr_c}\Bigg\{\cosh(\tilde{A}g_s s)\cos(\widetilde{B}g_s s)\times \\ & \times\frac{(\hat{n}\cdot\vec{\lambda}/2)^2}{|\mathrm{Det}(C)|}\int\mathrm{d}q^0\left\langle q^0\left|\exp\left[\mathrm{i}s|\alpha_1|\left(\widehat{P}^0\widehat{P}^0-\frac{g_s^2\tilde{A}^2}{|\alpha_1|^2}\widehat{X}^0\widehat{X}^0\right)\right]\right|q^0\right\rangle\times \\ & \times\int\mathrm{d}q^1\left\langle q^1\left|\exp\left[-\mathrm{i}s|\alpha_2|\left(\widehat{P}^1\widehat{P}^1+\frac{g_s^2\widetilde{B}^2}{|\alpha_2|^2}\widehat{X}^1\widehat{X}^1\right)\right]\right|q^1\right\rangle-\ldots\Bigg\}, \end{aligned}\tag{C.12}$$

where the dots represent all the time the analogous expression with zero fields. It is straightforward to check from (C.11), using expressions (B.12), (B.13) for ξ_4 and ξ_5, that the determinant $|\mathrm{Det}(C)|$ appearing in the denominator is equal to $1/|g_s^2\mathcal{E}\mathcal{B}_\parallel| = 1/|g_s^2\vec{\mathcal{E}}\cdot\vec{\mathcal{B}}| = 1/(g_s^2\tilde{a}\tilde{b})$ (cf. (2.12), (2.13) for definition of $\tilde{a}$ and $\tilde{b}$). Therefore, keeping in mind definitions (2.15) for matrices $\tilde{A}$ and $\widetilde{B}$, we see that the obtained expression (C.12) is in fact expression (2.17). This proves the implication (2.16) $\Rightarrow$ (2.17).

Appendix D.

Tracing over the harmonic-oscillator degrees of freedom

Here we calculate traces (integrals) of exponentiated harmonic oscillators appearing in expression (2.17), *i.e.* we calculate

$$T_1(s) = \int_{-\infty}^{+\infty}\mathrm{d}q^0\langle q^0|e^{\mathrm{i}s2|\alpha_1|\mathcal{H}^{(1)}}|q^0\rangle, \quad T_2(s) = \int_{-\infty}^{+\infty}\mathrm{d}q^1\langle q^1|e^{-\mathrm{i}s2|\alpha_2|\mathcal{H}^{(2)}}|q^1\rangle\,, \tag{D.1}$$

where $\mathcal{H}^{(k)}$ ($k = 1, 2$) are Hamiltonian densities of harmonic oscillators and have 3×3 color-matrix structure. They can be rewritten in a 3×3-diagonal form by applying unitary transformation (2.20)

$$\mathcal{H}^{(k)} = \mathcal{U}(\hat{n})\mathcal{H}^{(k)\mathrm{d}}\mathcal{U}^\dagger(\hat{n}) \quad (k = 1, 2)\,, \tag{D.2}$$

where $\mathcal{H}^{(k)\mathrm{d}}$ have a 3×3 diagonal matrix structure

$$\mathcal{H}^{(1)\mathrm{d}}_{ij} = \delta_{ij}\left\{\frac{1}{2}\widehat{P}^0\widehat{P}^0 - \frac{g_s^2\tilde{a}^2}{2|\alpha_1|^2}[(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}]^2\widehat{X}^0\widehat{X}^0\right\}, \tag{D.3}$$

$$\mathcal{H}^{(2)\mathrm{d}}_{ij} = \delta_{ij}\left\{\frac{1}{2}\widehat{P}^1\widehat{P}^1 + \frac{g_s^2\tilde{b}^2}{2|\alpha_2|^2}[(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}]^2\widehat{X}^1\widehat{X}^1\right\}. \tag{D.4}$$

Then trace integrals (D.1) can be rewritten as

$$T_1(s) = \mathcal{U}(\hat{n})\int_{-\infty}^{+\infty}\mathrm{d}q^0\langle q^0|\exp[\mathrm{i}s2|\alpha_1|\mathcal{H}^{(1)\mathrm{d}}]|q^0\rangle\mathcal{U}^\dagger(\hat{n}), \tag{D.5}$$

$$T_2(s) = \mathcal{U}(\hat{n})\int_{-\infty}^{+\infty}\mathrm{d}q^1\langle q^1|\exp[-\mathrm{i}s2|\alpha_2|\mathcal{H}^{(2)\mathrm{d}}]|q^1\rangle\mathcal{U}^\dagger(\hat{n}). \tag{D.6}$$

Expressions (D.3), (D.4) show that, in rotated basis, $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ are sets of three oscillators each, with frequency parameters $\omega = \mathrm{i}g_s\tilde{a}|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}|/|\alpha_1|$ and $\omega = g_s\tilde{b}|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}|/|\alpha_2|$ ($j = 1,2,3$), respectively, and with mass parameter $M = 1$. Since tracing (D.5), (D.6) for these (six) oscillators can be done in any complete basis, we choose instead of momentum-eigenstate basis $|q^0\rangle$ ($|q^1\rangle$) the basis of eigenstates of c-number Hamiltonian densities $\mathcal{H}^{(k)\mathrm{d}}_{jj}$ ($k = 1,2$; $j = 1,2,3$). Eigenenergies $\mathrm{i}E^{(1)}_j(m)$ (imaginary positive) of the first set, and $E^{(2)}_j(m)$ (real positive) of the second set of three harmonic Hamiltonians are then

$$\mathrm{i}E^{(1)}_j(m) = \mathrm{i}\frac{g_s\tilde{a}}{|\alpha_1|}|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}|\left(m+\frac{1}{2}\right),\ E^{(2)}_j(m) = \frac{g_s\tilde{b}}{|\alpha_2|}|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}|\left(m+\frac{1}{2}\right), \tag{D.7}$$

where $j(= 1,2,3)$ labels the oscillator, and $m = 0,1,2,\ldots$ are the energy quantum numbers of the harmonic oscillators. Tracing therefore reduces to the simple geometric sums

$$\begin{aligned}\int_{-\infty}^{+\infty}\mathrm{d}q^0\langle q^0|\exp[\mathrm{i}s2|\alpha_1|\mathcal{H}^{(1)\mathrm{d}}_{jj}]|q^0\rangle &= \sum_{m=0}^{\infty}\exp[\mathrm{i}s2|\alpha_1|\mathrm{i}E^{(1)}_j(m)] = \\ &= \exp[-g_ss\tilde{a}|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}|]\sum_{m=0}^{\infty}\{\exp[-2g_ss\tilde{a}|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}|]\}^m = \\ &= \frac{1}{2\sinh[g_ss\tilde{a}|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}|]},\end{aligned} \tag{D.8}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}\mathrm{d}q^1\langle q^1|\exp[-\mathrm{i}s2|\alpha_2|\mathcal{H}^{(2)\mathrm{d}}_{jj}]|q^1\rangle &= \sum_{m=0}^{\infty}\exp[-\mathrm{i}s2|\alpha_2|E^{(2)}_j(m)] = \\ &= \exp[-\mathrm{i}g_ss\tilde{b}|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}|]\sum_{m=0}^{\infty}\{\exp[-\mathrm{i}2g_ss\tilde{b}|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}|]\}^m =\end{aligned} \tag{D.9}$$

$$= \frac{1}{2\mathrm{i}\sin[g_s s\tilde{b}|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}|]} \,,$$

where no sum over j is taken. In order to ensure that the geometric sum in (D.9) converges, we have to move the proper time $s > 0$ slightly below the real axis: $s \mapsto z = s - \mathrm{i}\varepsilon'$ ($\varepsilon' = +0$). Rotating these expressions (3×3 diagonal matrices) back to the original color basis (cf. (D.5), (D.6)), we obtain the final result

$$T_1(s)T_2(s) = \frac{1}{4\mathrm{i}}\mathcal{U}(\hat{n})\frac{1}{\sinh[g_s z\tilde{a}|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}|]\sin[g_s z\tilde{b}|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}|]}\mathcal{U}^{\dagger}(\hat{n}) \tag{D.10}$$

$$= \frac{1}{4\mathrm{i}\sinh[g_s z\tilde{a}(\hat{n}\cdot\vec{\lambda}/2)]\sin[g_s z\tilde{b}(\hat{n}\cdot\vec{\lambda}/2)]} \,. \tag{D.11}$$

In (D.10) we denote by $|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}|$ the diagonal matrix whose diagonal elements are absolute values $|(\hat{n}\cdot\vec{\lambda}/2)^{\mathrm{d}}_{jj}|$. However, since both sinh and sin functions are odd under the change of sign of the argument, we can dispose of the absolute signs in (D.10); the indicated unitary rotation then leads to the final 3×3 matrix result (D.11), where absolute signs in the arguments-matrices were disposed of. Keeping in mind notation (D.1) for matrices $T_1(s)$ and $T_2(s)$, as well as notation (2.15), we see that the resulting formula (D.11) proves the implication (2.17) $\Rightarrow$ (2.18).

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