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Feynman quasiprobability distribution for spin-(1/2), and its generalizations(*)

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Summary. — We examine R. P. Feynman's paper "Negative probability", in which, after a discussion about the possibility of attributing a real physical meaning to quasiprobability distributions, he introduces a new kind of distribution for spin-(1/2), with a possible method of generalization to systems with arbitrary number of states. The principal aim of this article is to shed light upon the method of construction of these distributions, taking into consideration their application to some experiments, and discussing their positive and negative aspects.

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1. – Introduction

In one of his last papers [1], R. P. Feynman speaks about his reflections on the possibility of modifying one of the "tacit assumptions" which is usually made in the formulation of quantum field theory, namely that the probability belongs to the interval [0,1]. He explains that this modification was not usable for the original purpose which was to eliminate the divergences that appear in this theory. In spite of this, it can be useful for questions regarding the interpretation of non-relativistic quantum theory, that is to say for attributing a real physical meaning to quasiprobability distributions, introduced for the first time by E. Wigner [2], and widely used, in their numerous versions, for questions of various kind [3].

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In fact, as we will see, these distributions allow the results of quantum mechanics to be translated into the language of classical probability theory. This, in turn, permits classical logic to be used in the description of quantum systems, and contributes to making some old questions of quantum theory, such as non-locality of non-separable systems [4], less paradoxical.

Although the problem of interpretation of quasiprobability distribution has been tackled from innumerable points of view in the literature, with opinions both for [5, 6] and against [7, 8] the attribution of a direct physical meaning to the distributions, it seems that Feynman's way of tackling the problem is different from the preceding ones and can offer new ideas for reflection and discussion.

The introduction of negative probabilities is utilized by Feynman, in the schematization of a quantum system, by means of a classical system $\Re$(1). This system is characterized by a certain number of states S, and can be placed in conditions C, not directly accessible to observation. In this scheme the "observable" probability, belonging to the interval [0,1], of finding the system $\Re$, in one of its states is

$$\wp(S) = \sum_C \wp(C) \cdot \wp(S|C)\,, \tag{1.1}$$

with

$$\sum_C \wp(C) = 1 \quad \text{and} \quad \sum_S \wp(S|C) = 1\,, \tag{1.2}$$

where $\wp(S|C)$ is the probability of being in the state S, when the condition C holds. The probabilities on the right of (1.1), assume values which are not necessarily $\in [0, 1]$, and therefore they are not directly observable. However, they obey a constraint that makes the $\wp(S)$ always non-negative and correctly $\in [0, 1]$.

This schematization is accomplished by means of a particular representation of the density operator [9], that allows the quantities in the expression (1.1) to be constructed, so that the conditions C are represented by pairs of eigenstates of certain quantum observables, and the states S are represented by eigenstates of a generic Hermitian operator. In this way, $\wp(C)$ has properties similar to a quasiprobability distribution, while $\wp(S)$ is the quantum average value of the projection operator, relative to the eigenstate corresponding to S.

This schematization also shows that there is a connection between the constraint imposed to ensure the positivity of the physical probability, and the uncertainty principle. This connection is due to the fact that the constraint prevents the appearance of negative probabilities in the physical states, and confines them to non-observable conditions (in which non-commuting observables are simultaneously defined) that are experimentally inaccessible also in quantum mechanics, because of the uncertainty principle.

We point out to the reader that the appearance of "virtual" states, which are not directly accessible to observation, is not completely new. Consider, for example, the virtual states (ghosts), of the electromagnetic field, in the covariant quantization procedure of Gupta and Bleuler [10, 11]. These states, non-observable by virtue of a mathematical constraint, turn out to be associated with a probability which does not always belong to [0,1], and with a non-physical polarization.

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(1) Feynman chooses a roulette wheel as classical system, to focus ideas.

Furthermore, it is worth noting that the search for an alternative representation of the density operator is justified by the observations of many important authors. We mention for example P. A. M. Dirac, who says about this operator [9]: "Its existence is rather surprising in view of the fact that phase space has no meaning in quantum mechanics, there being no possibility of assigning numerical values simultaneously to the q's and p's". Therefore, finding a representation of the density operator in order to allow a stochastic interpretation, would somehow justify its existence, and permits something like the classical phase space to be constructed for quantum systems too.

The main aim of this article is therefore to clarify and to generalize whenever possible, the method for constructing distributions which Feynman proposes in "Negative probability" [1]. Among the applications, we will consider the EPRB [12] and GHZ [13] experiments that will show that the distribution scheme is similar to a local hidden-variable theory [4], and not in contradiction with the results of quantum mechanics. A critical discussion at the end of the paper will seek to evaluate the various aspect of the method itself.

2. – Feynman distribution for spin-(1/2)

After having discussed the interpretation of negative probability, Feynman introduces some operators whose expectation values, he says, are an analogue of the Wigner function for a spin-(1/2) system.

The expressions of these operators are the following:

$$\begin{aligned} \hat{f}_{++} &= \frac{1}{4}(1+\hat{\sigma}_z+\hat{\sigma}_x+\hat{\sigma}_y)\,, \\ \hat{f}_{+-} &= \frac{1}{4}(1+\hat{\sigma}_z-\hat{\sigma}_x-\hat{\sigma}_y)\,, \\ \hat{f}_{-+} &= \frac{1}{4}(1-\hat{\sigma}_z+\hat{\sigma}_x-\hat{\sigma}_y)\,, \\ \hat{f}_{--} &= \frac{1}{4}(1-\hat{\sigma}_z-\hat{\sigma}_x+\hat{\sigma}_y)\,, \end{aligned} \tag{2.1}$$

where $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ are the Pauli matrices, and $f_{++} = \mathrm{tr}\{\hat{\rho}\cdot\hat{f}_{++}\}$, and so on, $\hat{\rho}$ being the density operator. From now on the operators will be indicated by the symbol ^, and sometimes for writing convenience we will set $f_{ji} = f(i,j)$, indicating by j the eigenvalues of the spin along the z-axis, and by i the eigenvalues of the spin along the x-axis. The average values of the operators (2.1) represent the probabilities of having the spin components, along the x-axis, and the z-axis, with simultaneously determinate values. As one can easily verify, these probabilities do not always belong to the interval [0,1]. This is in agreement with the fact that they are non-directly observable, since it is impossible to measure the two spin components at the same time (owing to the uncertainty principle). Furthermore, Feynman gives the probability of having the spin pointing up in a given direction V, if one of the conditions (i,j) holds (with $i,j=+,-$), defined as

$$\Pi_{V+}(i,j) = \frac{1}{2}(1+jV_z+iV_x+i\cdot jV_y)\,. \tag{2.2}$$

Here (V_x, V_y, V_z) are the Cartesian components of the unit vector in the considered direction.

Note that quantities (2.1) and (2.2) allow the following expression to be written:

$$\mathrm{tr}\{\hat{\rho} \cdot \widehat{\Pi}_{V+}\} = \sum_{i,j} f(i,j) \cdot \Pi_{V+}(i,j)\,, \tag{2.3}$$

$\widehat{\Pi}_{V+}$ being the spin projection operator relative to the up-state along the direction V. The expression (2.3) is easy to prove, using the fact that the operators (2.1) are a basis in the 2×2 Hermitian matrix space.

Observe that (2.3) is formally similar to (1.1), the conditions (i,j) being the analogue of the conditions C, and the spin eigenstates in the direction V, the analogue of the states S. The relations (1.2), as one can immediately verify, are also fulfilled.

Naturally, the fundamental problem is understanding how the operators (2.1) are defined (and consequently also the "weights" (2.2)), which in turn permits the distribution to be generalized to an arbitrary system. The derivation of operators (2.1) has already been object of investigation [14], but in our opinion in this case the derivation did not completely correspond to Feynman's ideas.

The definition of (2.1) may be deduced from the final part of Feynman's paper, where he gives a brief explanation of the method generalization, and so also for constructing operators.

Feynman examines a quantum event that may happen in two ways, to which the amplitudes a and b, respectively, are associated, and introduces a quantity that he calls "probability" of that event "coming" in way A and "going" in way B (or "looping" via A and B), defined as

$$P(A,B) = \mathrm{Re}[(1+\iota)a^* b] = \frac{(1+\iota)}{2}a^* b + \frac{(1-\iota)}{2}ab^* \tag{2.4}$$

(we set $\iota = \sqrt{-1}$). He adds furthermore that the usual probability of an event happening in many ways A, B, C, etc. with amplitudes a, b, c,..., can be written as a sum of "probabilities" constructed in the same way as (2.4). Considering, for the sake of simplicity, only the ways A and B, we have

$$\begin{aligned}(a^* + b^*)(a+b) &= P(A,A) + P(A,B) + P(B,A) + P(B,B) = \\ &= \mathrm{Re}[(1+\iota)a^* a] + \mathrm{Re}[(1+\iota)a^* b] + \mathrm{Re}[(1+\iota)ab^*] + \mathrm{Re}[(1+\iota)b^* b]\,.\end{aligned} \tag{2.5}$$

Of course, the "probability" $P(A,A)$ has only one way of being constructed, that is $P(A,A) = \mathrm{Re}[(1+\iota)a^* a] = a^* a$, equal to the conventional positive probability of event A happening. This "probability", if the two ways of "coming" and "going" are not the same, is not necessarily a non-negative quantity. In fact, if $a \neq b$, then clearly the $P(A,B)$ and $P(B,A)$ can be both positive and negative. Quoting Feynman's precise words: "The total probability is the sum of these P for every pair of ways. If the two ways in P "coming" and "going" are not the same, P is as likely to be negative as positive".

According to this definition, we can construct, for example, the "probability", "coming" in x up, and "going" in z up, considering a spin state $|\Psi\rangle$, and taking $a = \langle x + |\Psi\rangle$ and $b = \langle z + |\Psi\rangle$, obtaining in this way

$$\begin{aligned}\mathrm{Re}[(1+\iota)\langle\Psi|x+\rangle\langle z+|\Psi\rangle] &= \langle\Psi|\mathrm{Re}[(1+\iota)|x+\rangle\langle z+|]\,|\Psi\rangle = \\ &= \langle\Psi|\frac{\sqrt{2}}{4}(1+\hat{\sigma}_z+\hat{\sigma}_x+\hat{\sigma}_y)|\Psi\rangle =\end{aligned} \tag{2.6}$$

$$= \operatorname{tr}\left\{\frac{\sqrt{2}}{4}(1+\hat{\sigma}_z+\hat{\sigma}_x+\hat{\sigma}_y)\cdot\hat{\rho}\right\}.$$

Here $|zj\rangle$ and $|xi\rangle$ are the eigenstates of the Pauli matrices in their usual representation, that is

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Expression (2.6) is also valid in the case of quantum mixtures. As can be seen, the operator constructed in (2.6) corresponds to the first of (2.1), except for a normalization factor. In the same way we can also obtain the remaining operators in (2.1). Furthermore, it is easy to see that

$$\operatorname{Re}[(1+\imath)\langle V+|xi\rangle\langle zj|V+\rangle] = \pm\frac{\sqrt{2}}{4}(1+jV_z+iV_x+i\cdot jV_y) \tag{2.7}$$

coincides with (2.2), except for a normalization factor.

This factor may be calculated adding (2.7) to the analogous quantity, constructed for $|V-\rangle$, that is, summing over all possible states:

$$\begin{aligned} v_{ji} &= \operatorname{Re}[(1+\imath)\langle V+|xi\rangle\langle zj|V+\rangle] + \operatorname{Re}[(1+\imath)\langle V-|xi\rangle\langle zj|V-\rangle] = \\ &= \operatorname{Re}[(1+\imath)\langle zj|xi\rangle] = \pm 1/\sqrt{2}\,. \end{aligned} \tag{2.8}$$

From the physical point of view, this means that (2.7) are interpreted as relative probabilities, and therefore normalized in the usual way

$$\Pi_{V+}(i,j) = \operatorname{Re}[(1+\imath)\langle V+|xi\rangle\langle zj|V+\rangle]\cdot v_{ji}^{-1}\,. \tag{2.9}$$

The factor (2.8), in addition to normalizing the probability (2.7), allows the quantities f_{++}, and so on, to be obtained, starting from the expression (2.6), and from other similar expressions. In fact, the operators are

$$\hat{f}_{ji} = \operatorname{Re}[(1+\imath)|xi\rangle\langle zj|]\cdot v_{ji}\,. \tag{2.10}$$

At this point it is natural that the reader should ask some questions about this procedure.

For example it is natural to ask why the decomposition (2.5) must be done in precisely that manner, or why one must use the operators $\hat{\sigma}_x$, and $\hat{\sigma}_z$, to construct the conditions (i,j), instead of two spin operators in any direction. Finally, the choice of one among the two possible directions of "looping", also remains without apparent explanation.

A possible answer to the first question is that the decomposition (2.5) has a heuristic value, and therefore the construction of real quantities similar to those in (2.4) and their use in representing the density matrix, may appear reasonable only after the completion of the general framework. Concerning the second question, as we will see shortly, the answer is that Feynman's method provides the use particular non-commuting operators for the construction of distributions. These operators must satisfy certain conditions, that represent the mathematical constraint connected with the uncertainty principle, which we talked about in the introduction. Finally, the third question about the two possible directions of "looping" remains without an explanation, although it might be justified simply by saying that there are two possible equivalent representations.

We now move on to discuss the generalization of the method, trying, as far as possible, to clarify the questions we have just mentioned.

3. – Generalization of the method

Distribution for n-state systems

Let us return for a moment to Feynman's text: "The density matrix ρ_{ij},. . . is then represented instead by saying a system has a probability to be found in each of a set of conditions. These conditions are defined by an ordered pair of states "coming" in i and "going" in j, with a "probability" $p(i,j)$ equal to the real part of $(1+\iota)\rho_{ij}$". This description corresponds to that which we obtained for spin-(1/2). In fact the result

$$p(i,j) = \mathrm{Re}[(1+\iota)\langle j|\hat{\rho}|i\rangle] = \mathrm{tr}\{\hat{p}(i,j)\cdot\hat{\rho}\}\,, \tag{3.1a}$$

where we set

$$\hat{p}(i,j) = \mathrm{Re}[(1+\iota)|i\rangle\langle j|]\,, \tag{3.1b}$$

corresponds to that already obtained, if we choose $|zj\rangle = |j\rangle$ and $|xi\rangle = |i\rangle$ (in general $|i\rangle$ and $|j\rangle$ are eigenstates of non-commuting observables of the system under consideration, that we denote by $\hat{A}$ and $\widehat{B}$). The weight of a projection operator $\widehat{\Pi}_s = |s\rangle\langle s|$, on an eigenstate(2) of a Hermitian operator is defined as

$$\mathrm{tr}\{\hat{p}(i,j)\cdot\widehat{\Pi}_s\} \tag{3.2}$$

and the normalization factor turns out to be

$$v_{ji} = \sum_s \mathrm{tr}\{\hat{p}(i,j)\cdot\widehat{\Pi}_s\} = \mathrm{tr}\{\hat{p}(i,j)\}\,. \tag{3.3}$$

We leave aside for a moment the discussion of the normalization factor, it will be re-examined later. First we will reason directly in terms of non-normalized expressions $p(i,j)$, as Feynman does. In general, the observables $\widehat{A}$ and $\widehat{B}$, which we assume to be non-commuting, must obey a constraint. This constraint, which is related to the uncertainty principle, renders the physical probability always positive, and corresponding to the correct value which is expected in quantum mechanics. We want to show that the sum over the conditions (i,j) of the products of (3.1a) and (3.2) produces the correct quantum results, *i.e.*

$$\mathrm{tr}\{\hat{\rho}\cdot\widehat{\Pi}_s\}\,,$$

if the constraint, linked to the uncertainty principle, is satisfied by the $p(i,j)$.

Feynman describes this constraint in words, both in the spin-(1/2) case and in the general one. It can be demonstrated that the constraint given by Feynman is satisfied if the $\hat{p}(i,j)$ are a basis, in a suitable Hermitian operator space(3). This space is endowed

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(2) For details on observables with a degenerate spectrum see below.
(3) For details about this space see below.

with a scalar product defined as the trace of a two-operator product, so that $\hat{p}(i,j)$ satisfy an "orthonormality" relation:

$$\mathrm{tr}\{\hat{p}(i,j)\cdot\hat{p}(m,n)\} = \delta_{im}\cdot\delta_{jn} \tag{3.4a}$$

and in the infinite-dimensional case an additional "completeness" condition. We refer the reader to the appendix for a proof that Feynman's constraint is satisfied, if the $\hat{p}(i,j)$ are an orthonormal and complete system. It can easily be shown that the condition (3.4a) is equivalent to

$$\langle i|j\rangle = \langle j|i\rangle \quad \forall\, i,j \quad \text{or} \quad \langle i|j\rangle = -\langle j|i\rangle \quad \forall\, i,j\,. \tag{3.4b}$$

Therefore, we must use particular non-commuting operators, which have eigenstates satisfying (3.4b), once the arbitrary phase factors are appropriately fixed. Clearly (3.4b) are equivalent since the one can be obtained from the other by multiplying it by a phase factor.

Note that the orthonormality and completeness conditions are very important, not only to expand every observable in terms of $\hat{p}(i,j)$, but also to satisfy other conditions, such as, in time evolution, the reduction of the transition probability, when $\Delta t \to 0$, to the Kronecker delta (see below). Therefore, it is necessary to adopt these two conditions as a constraint, to ensure that the procedure is consistent.

If we consider the case of operators with a finite-dimensional spectrum of dimension d (the infinite-dimensional case will be considered later), the constraint (3.4a) makes the operators $\hat{p}(i,j) = \mathrm{Re}[(1+\imath)|i\rangle\langle j|]$ an orthonormal basis in the vector space over the real field, of $d\times d$ Hermitian matrices, in which the scalar product is the trace of the two-matrix product. It furnishes, therefore, the possibility of developing any Hermitian operator $\widehat{O}$ (and so also a projector) in this way

$$\widehat{O} = \sum_{ij}\hat{p}(i,j)\cdot\mathrm{tr}\{\hat{p}(i,j)\cdot\widehat{O}\}\,, \tag{3.5a}$$

and also of having

$$\mathrm{tr}\{\widehat{O}\cdot\hat{\rho}\} = \sum_{ij}\mathrm{tr}\{\hat{p}(i,j)\cdot\hat{\rho}\}\cdot\mathrm{tr}\{\hat{p}(i,j)\cdot\widehat{O}\}\,. \tag{3.5b}$$

Observing carefully the relations (3.4b), one can verify that, if the operators $\hat{A}$ and $\widehat{B}$ commute, then, since they have simultaneous eigenstates, the first of the relations is automatically satisfied:

$$\langle i|j\rangle = \delta_{ij} = \langle j|i\rangle\,. \tag{3.6}$$

Therefore, the constraint is connected with the uncertainty principle, since it is automatically satisfied if the operators $\hat{A}$ and $\widehat{B}$ commute. For this reason the constraint must be imposed when the operators do not commute, to avoid the appearance of probabilities which do not belong to [0,1], in the physical states.

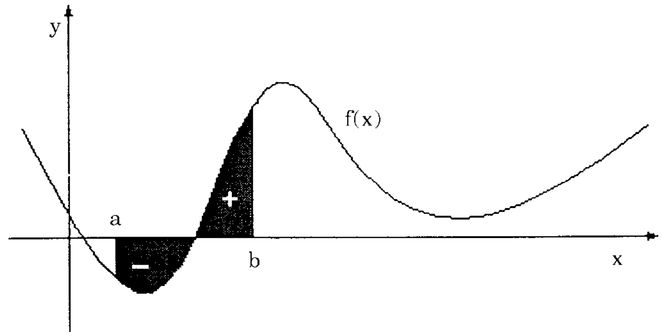

Fig. 1. – $\int_a^b f(x)\mathrm{d}x = 0$. $f(x)$ = probability density with negative values.

Now the problem of normalization needs to be discussed. Clearly, if $v_{ji} \neq 0$, the problem does not exist, and the normalized quantities are defined as they were for spin-(1/2):

$$\hat{f}_{ji} = \hat{f}(i,j) = \hat{p}(i,j) \cdot v_{ji}\,, \tag{3.7a}$$

$$\Pi_s(i,j) = \mathrm{tr}\{\hat{p}(i,j) \cdot \widehat{\Pi}_s\} \cdot v_{ji}^{-1}\,. \tag{3.7b}$$

In this way the distribution $f(i,j) = \mathrm{tr}\{\hat{f}(i,j) \cdot \hat{\rho}\}$ has the properties of a Wigner-like distribution, and furthermore (3.7) satisfies relations analogous to (1.1) and (1.2). It is clear that, in this case, the weight of a generic Hermitian operator

$$O(i,j) = \mathrm{tr}\{\hat{p}(i,j) \cdot \widehat{O}\} v_{ji}^{-1} = \sum_s o_s \mathrm{tr}\{\hat{p}(i,j) \cdot \widehat{\Pi}_s\} v_{ji}^{-1}$$

(with $\widehat{O}|s\rangle = o_s|s\rangle$)([4]) can be interpreted as the average value of the physical quantity corresponding to the operator, when the condition (i,j) holds. This average is made over the s-states , wherein the quantity may assume values equal to eigenvalues o_s.

On the contrary, if $v_{ji} = 0$, there are some problems. Of course these problems not only regard the Feynman distribution, but also regard quasiprobability distributions in general. In fact, if we have a probability distribution which also has negative values, there are some subsets of the stochastic variable definition set that have an associated null probability. Therefore, if we want to consider the conditional probabilities associated with such subsets, the normalization gives rise to undesired zeroes in the denominators (see fig. 1).

In these cases, in which the normalization factor is zero, Feynman's suggestion is to operate with non-normalized quantities. Indeed, this is confirmed by the fact that, in one of his examples (that of the particle diffusion), he shows that sometimes one

([4]) If the spectrum of $\widehat{O}$ contains degenerate eigenvalues, then, when a projector which corresponds to an eigenvalue of this kind is considered in (3.2), it must be interpreted as a projector on an eigenspace generated by eigenvectors corresponding to the degenerate eigenvalue. Furthermore, if some of the eigenvalues of $\hat{A}$ and $\widehat{B}$ are degenerate, it is advantageous to label the conditions (i,j), considering eigenstates, rather than eigenvalues.

can also reason in terms of non-normalized quantities. Furthermore, the fact that in the generalization he reasons directly in terms of $p(i,j)$ again confirms that he does not express the normalization factor explicitly, in order to avoid running up against expressions with a zero in the denominator.

From now on we will use normalized quantities, with the warning that, when $v_{ji}=0$, quantities are to be considered non-normalized.

Composite system distribution

For a system composed of several elementary subsystems, the distribution is defined setting

$$\hat{p}(i,j)=\bigotimes_{\alpha=1}^{N}\hat{p}^{(\alpha)}(i_\alpha,j_\alpha)\,, \tag{3.8}$$

where $\hat{p}^{(\alpha)}(i_\alpha,j_\alpha)$ are operators defined as in (3.1b) of the single subsystem.

Note that the definition is the same both for identical and differing subsystems. The weight of a Hermitian operator, and the normalization factor, can be deduced from (3.8), in the same way as in (3.5a) and (3.3). Clearly if the orthonormality property holds for the elementary subsystems, it continues to hold for the composite system. In fact,

$$\text{(3.9)}\quad \mathrm{tr}\left\{\bigotimes_{\alpha=1}^{N}\hat{p}^{(\alpha)}(i_\alpha,j_\alpha)\cdot\bigotimes_{\beta=1}^{N}\hat{p}^{(\beta)}(m_\beta,n_\beta)\right\}=\mathrm{tr}\left\{\bigotimes_{\alpha=1}^{N}\hat{p}^{(\alpha)}(i_\alpha,j_\alpha)\cdot\hat{p}^{(\alpha)}(m_\alpha,n_\alpha)\right\}=$$
$$=\prod_{\alpha=1}^{N}\mathrm{tr}\{\hat{p}^{(\alpha)}(i_\alpha,j_\alpha)\cdot\hat{p}^{(\alpha)}(m_\alpha,n_\alpha)\}=\prod_{\alpha=1}^{N}\delta_{i_\alpha m_\alpha}\cdot\delta_{j_\alpha n_\alpha}\,.$$

Below, when we consider the two-particle spin-(1/2) system, we will verify that the definition (3.8) exactly reproduces Feynman's results, and therefore constitutes a generalization of the latter.

Time evolution

It can be shown, using the properties (3.4a) and (3.9), that the following relations hold (in the case of composite systems we assume $i=i_1,i_2,\dots,i_N$, etc.):

$$f(m,n,t)=\sum_{i,j}K(m,n,t|i,j,t_0)\cdot f(i,j,t_0)\,, \tag{3.10a}$$

$$K(m,n,t|g,h,t_0)=\sum_{i,j}K(m,n,t|i,j,t')\cdot K(i,j,t'|g,h,t_0)\,, \tag{3.10b}$$

being $f(i,j,t)=\mathrm{tr}\{\widehat{S}^*\hat{f}_{ji}\widehat{S}\cdot\hat{\rho}\}$, and

$$K(m,n,t|i,j,t_0)=\mathrm{tr}\{\hat{p}(i,j,t_0)\widehat{S}^*\hat{p}(m,n,t_0)\widehat{S}\}\cdot v_{nm}/v_{ji}\,, \tag{3.11}$$

where we set $|i,t_0\rangle = |i\rangle$, $|j,t_0\rangle = |j\rangle$, in $\hat{p}(i,j)$, etc., and $\widehat{S}$ is the unitary operator of quantum mechanics time evolution. Moreover,

$$\sum_{mn} K(m,n,t|i,j,t_0) = 1\,, \tag{3.12a}$$

$$\lim_{t\to t_0} K(m,n,t|i,j,t_0) = \delta_{mi}\cdot\delta_{nj}\,. \tag{3.12b}$$

It is clear that, the expressions (3.10) make the time evolution of the f's similar to that of a Markov stochastic process. Equations (3.12) justify the interpretation of $K(m,n,t|i,j,t_0)$ as a transition probability from the state (i,j) to (m,n). It is not difficult to verify that the transition probability (3.11) corresponds to the expression given by Feynman for this quantity (remember that Feynman operates with non-normalized probability $p(i,j)$), that is, in formulas

$$\mathrm{tr}\{\hat{p}(i,j,t_0)\widehat{S}^*\hat{p}(m,n,t_0)\widehat{S}\} = \frac{1}{2}[\widehat{S}^*_{im}\widehat{S}_{jn} + \widehat{S}^*_{jn}\widehat{S}_{im}] + \frac{\iota}{2}[\widehat{S}^*_{jm}\widehat{S}_{in} - \widehat{S}^*_{in}\widehat{S}_{jm}]$$

(with $\widehat{S}_{im} = \langle m|\widehat{S}|i\rangle$ and $\widehat{S}^*_{im} = (\langle m|\widehat{S}|i\rangle)^* = \langle i|\widehat{S}^*|m\rangle$).

Infinite-dimensional systems

We now discuss the case of the operators $\widehat{A}$ and $\widehat{B}$ operating in an infinite-dimensional space.

When operators $\widehat{A}$ and $\widehat{B}$ have a discrete spectrum and are endowed with a complete set of eigenstates, the operators $\hat{p}(i,j) = \mathrm{Re}[(1+\iota)|i\rangle\langle j|]$ satisfy a relation identical to (3.4a), if the eigenstates can be chosen to satisfy (3.4b). In this case, the considerations made for finite-dimensional operators can be repeated unchanged, if we consider Hilbert-Schmidt Hermitian([5]) operators [15], *i.e.* so that

$$\mathrm{tr}\{\widehat{O}^*\cdot\widehat{O}\} < \infty\,. \tag{3.13}$$

The operators $\hat{p}(i,j) = \mathrm{Re}[(1+\iota)|i\rangle\langle j|]$ are an orthonormal basis in the vector space of these operators, that form a Hilbert space **H** if endowed with a scalar product defined as

$$\mathrm{tr}\{\widehat{A}^*\cdot\widehat{B}\}\,. \tag{3.14}$$

Since only Hermitian operators are considered, the Hilbert space is realized over the real field. Furthermore, it is easy to show, starting from the relations (3.13) and (3.5a), that the sequence $\xi(i,j) = \mathrm{tr}\{\hat{p}(i,j)\cdot\widehat{O}\}$ is square-summable, and that the series which is a generalization of the sum (3.5b) is convergent, because $\hat{\rho}$ always satisfies (3.13), since $\mathrm{tr}\{\hat{\rho}^2\}\le 1$. It is easy to see that every Hilbert-Schmidt Hermitian operator, except for the null operator, satisfies the relation

$$\mathrm{tr}\{\hat{p}(i,j)\cdot\widehat{O}\}\neq 0$$

with at least one of the $\hat{p}(i,j)$, if $|j\rangle$ and $|i\rangle$ are complete bases in the Hilbert space. This confirms that the $\hat{p}(i,j)$ are a complete basis. In addition to this, observe that, because

([5]) In this article Hermitian is synonymous with self-adjoint.

$\hat{\rho}$ is trace class, if $\widehat{O}$ is bounded (*i.e.* $\sup_{\||\Psi\rangle\|\neq 0} \|\widehat{O}|\Psi\rangle\|/\||\Psi\rangle\| < \infty$ if $|\Psi\rangle \in \mathbf{H}$), we have [15]

$$|\mathrm{tr}\{\hat{\rho}\cdot\widehat{O}\}| < \infty\,,$$

in this way, we can obtain

$$|\mathrm{tr}\{\hat{\rho}\cdot\widehat{O}\}| = |\sum_{ij}\mathrm{tr}\{\hat{\rho}\cdot\hat{p}\}\cdot\mathrm{tr}\{\widehat{O}\cdot\hat{p}\}| < \infty \tag{3.15}$$

($\hat{p} = \hat{p}(i,j)$), that is to say the expansion (3.15) is also convergent for a bounded operator. The expression (3.15) furnishes the possibility of developing the expression $\mathrm{tr}\{\hat{\rho}\cdot\widehat{\Pi}_{\widehat{O}}(E)\}$, where $\widehat{\Pi}_{\widehat{O}}(E)$ is an element of the p. v. m. (projection-valued measure), of the generic operator $\widehat{O}$, relative to the Borel set E, since $\widehat{\Pi}_{\widehat{O}}(E)$ is bounded. This permits the analogue of (1.1) to be written for a quantum system, for the spectrum of any operator.

In the even more general case of operators $\hat{A}$ and $\widehat{B}$, with pieces of continuous spectrum, when in the expression (3.4a) at least one of the two Kronecker delta is replaced by a Dirac delta, (3.13) is no longer satisfied. We will not attempt to demonstrate, in this case, the convergence of expansion similar to (3.5a) (with the sum replaced by an integral for the continuous pieces of the spectrum) because this is outside the scope of this article.

4. – Applications

EPRB

The first application we will consider is the EPRB [12] experiment, showing that the definition of composite system distribution given in (3.8) reproduces the distribution that Feynman gives numerically only in the case of two spin-(1/2) particles, in a singlet state. For a detailed discussion of the experiment, we refer the reader to the paper "Negative probability" [1].

We can easily check that quantity

$$f(i_1,j_1,i_2,j_2) = \langle\Psi_0|\hat{f}^{(1)}(i_1,j_1)\otimes\hat{f}^{(2)}(i_2,j_2)|\Psi_0\rangle\,, \tag{4.1}$$

where $|\Psi_0\rangle$ is the spin singlet, reproduces the values given by Feynman: $+1/8$ if $(i_1,j_1)\neq(i_2,j_2)$ and $-1/8$ if $(i_1,j_1)=(i_2,j_2)$. Furthermore the quantity

$$\mathrm{tr}\{\hat{p}^{(1)}\otimes\hat{p}^{(2)}\cdot\widehat{\Pi}^{(1)}_{V+}\otimes\widehat{\Pi}^{(2)}_{U+}\}/\mathrm{tr}\{\hat{p}^{(1)}\otimes\hat{p}^{(2)}\}\,, \tag{4.2}$$

where $\hat{p}^{(1)} = \hat{p}^{(1)}(i_1,j_1)$ and $\widehat{\Pi}^{(1)}_{V+} = |V+\rangle\langle V+|$ etc..., coincides with the product of the weights, defined in the case of a single system

$$\Pi_{V+}(i_1,j_1)\cdot\Pi_{U+}(i_2,j_2)\,, \tag{4.3}$$

exactly in accord with Feynman's words.

We can conclude that, using this schematization, it is possible to construct a model similar to a local hidden-variable theory [4], as Feynman himself shows in the cited text [1].

GHZ

Before passing on to an examination of GHZ, we will make a brief excursion to discuss the meaning of "element of reality" in the Feynman schematization, considering for simplicity's sake the spin-(1/2) case.

In accordance with what was said in the preceding section, if we consider the quantities $\sigma_V(i,j) = \sum_s s \cdot \Pi_{Vs}(i,j)$, (where $s = +, -$ are the $\hat{\sigma}_V$ eigenvalues), they can be considered to be the average values of the spin along the V axis, when the condition (i,j) holds. For this reason, if

$$\Pi_{Vs}(i,j) = \delta_{ss'}\,, \tag{4.4}$$

then

$$\sigma_V(i,j) = s' \tag{4.5}$$

can be interpreted as an element of reality that the system possesses when the condition (i,j) holds. Obviously this interpretation implies giving to the non-observable probability (4.4), the same meaning as if it were a physical probability, that is: $1 =$ certainty of the event, $0 =$ impossibility of the event. However, the probability of being in the condition (i,j) does not always belong to [0,1], and so it is not directly observable. Therefore, the interpretation we have given to (4.5), which seems to have a formal meaning, shows that in this schematization the elements of reality are associated with a non-directly measurable probability.

Now we shall see how, what we have just said can be applied to an experiment that may be considered as an extension of EPRB to a three-spin system, that is the GHZ "gedanken" experiment [13]. Let us give a brief reminder of this experiment.

Consider three spin-(1/2) particles, produced in a source decay. For the sake of simplicity, suppose that their trajectories are complanate. For every particle choose a reference frame, with the z-axis oriented along the trajectory in the direction of motion, the x-axis orthogonal to the motion plane, and the y-axis orthogonal to the trajectory. A set of mutually commuting operators that describe completely the three-particle system is the following:

$$\hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_y^{(2)} \otimes \hat{\sigma}_y^{(3)}, \quad \hat{\sigma}_y^{(1)} \otimes \hat{\sigma}_x^{(2)} \otimes \hat{\sigma}_y^{(3)}, \quad \hat{\sigma}_y^{(1)} \otimes \hat{\sigma}_y^{(2)} \otimes \hat{\sigma}_x^{(3)}\,, \tag{4.6}$$

where the operators $\hat{\sigma}_x^{(\alpha)}$ and $\hat{\sigma}_y^{(\alpha)}$ are spin operators corresponding to the respective axes of the α-th particle.

Suppose that the particles are in the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|+,+,+\rangle - |-,-,-\rangle)\,, \tag{4.7}$$

being

$$|\pm,\pm,\pm\rangle = |z,\pm\rangle^{(1)} \otimes |z,\pm\rangle^{(2)} \otimes |z,\pm\rangle^{(3)}\,.$$

It can be shown that the hypothesis that the elements of reality, corresponding to the three spin components for each particle, exist, can be logically deduced from the fact that operators (4.6) have the state (4.7) as an eigenstate, with eigenvalue $+1$. This leads to a contradiction with quantum mechanics, because it predicts that the product of the measures of three particle spin components along the x-axis, will have the value $+1$, while the quantum value is -1. We refer the reader to ref. [13], for a complete description of the experiment.

As in the EPRB experiment, in the three-particle case too, we can define the analogue of the Wigner function, making the quantum mean value of tensor product of the Feynman operators, in the state (4.7):

$$f(i_1, j_1, i_2, j_2, i_3, j_3) = \langle \Psi | \hat{f}^{(1)}(i_1, j_1) \otimes \hat{f}^{(2)}(i_2, j_2) \otimes \hat{f}^{(3)}(i_3, j_3) | \Psi \rangle \,. \tag{4.8}$$

In this way the mean value of the three operators in quantum state (4.7) can be written as a weighted average with respect to the function (4.8), that is to say

$$\begin{aligned} &\langle \Psi | \hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_y^{(2)} \otimes \hat{\sigma}_y^{(3)} | \Psi \rangle = \\ &= \sum_{ij} \langle \Psi | \hat{f}^{(1)}(i_1, j_1) \otimes \hat{f}^{(2)}(i_2, j_2) \otimes \hat{f}^{(3)}(i_3, j_3) | \Psi \rangle \sigma_x^{(1)}(i_1, j_1) \sigma_y^{(2)}(i_2, j_2) \sigma_y^{(3)}(i_3, j_3) \,. \end{aligned}$$

(We set: $i = i_1, i_2, i_3$ and $j = j_1, j_2, j_3$.) In a similar fashion, this procedure can be performed for the other operators (4.6). We see therefore that in this three-particle system, the elements of reality, corresponding to spin components along the x- and y-axes, exist, and are associated with (i, j) conditions. In fact, according to the interpretation we have just made (formula (4.5)), we can assert that the spin-(1/2) systems possess elements of reality, corresponding to spin components along the x-, y- and z-axes, given by weights $\sigma_x(i, j)$, $\sigma_y(i, j)$ and $\sigma_z(i, j)$.

It is clear that, the products $\sigma_x^{(1)} \cdot \sigma_y^{(2)} \cdot \sigma_y^{(3)}$, $\sigma_y^{(1)} \cdot \sigma_x^{(2)} \cdot \sigma_y^{(3)}$ and $\sigma_y^{(1)} \cdot \sigma_y^{(2)} \cdot \sigma_x^{(3)}$ (where $\sigma_x^{(1)}(i_1, j_1) = \sigma_x^{(1)}$ and so on) do not always equal $+1$, but equal $+1$ in some conditions, and -1 in others. This avoids the GHZ contradiction, naturally at the cost of introducing states that have an occupation probability, that can also be negative. In fact the product $\sigma_x^{(1)} \cdot \sigma_x^{(2)} \cdot \sigma_x^{(3)}$ has value $+1$ in certain conditions and -1 in others, so that the Wigner-like function (4.8), defined for the state (4.7), can furnish the value of the weighted average coinciding with the quantum value -1:

$$\begin{aligned} &-1 = \langle \Psi | \hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_x^{(2)} \otimes \hat{\sigma}_x^{(3)} | \Psi \rangle = \\ &= \sum_{ij} \langle \Psi | \hat{f}^{(1)}(i_1, j_1) \otimes \hat{f}^{(2)}(i_2, j_2) \otimes \hat{f}^{(3)}(i_3, j_3) | \Psi \rangle \sigma_x^{(1)}(i_1, j_1) \sigma_x^{(2)}(i_2, j_2) \sigma_x^{(3)}(i_3, j_3) \,. \end{aligned}$$

In conclusion, we can assert that in the GHZ case too, the analogue of the Wigner function for spin-(1/2) allows for a realistic model, although with the introduction of "virtual" states which are not directly observable, wherein the system fluctuation is described by a probability not always $\in [0, 1]$.

Other applications

We will now show that for the harmonic oscillator, it is not difficult to find two non-commuting operators obeying the constraint (3.4). In this case, we can choose, for example, the "number" operator

$$\hat{n} = \hat{a}^* \hat{a} \tag{4.9}$$

(where $\hat{a}^*$ and $\hat{a}$ are the creation and destruction operators) and the operator [16]

$$\widehat{\cos}\Phi = \frac{1}{2}[\widehat{\exp}[\iota\Phi] + \widehat{\exp}[-\iota\Phi]]\,, \tag{4.10}$$

being

$$[\widehat{\cos}\Phi, \hat{n}] = \iota\,\widehat{\sin}\Phi, \quad \widehat{\cos}\Phi|\cos\Phi\rangle = \cos\Phi|\cos\Phi\rangle, \quad 0 \le \Phi \le \pi\,.$$

Therefore the Feynman distribution is constructed in the following way:

$$p(\cos\Phi, n) = \mathrm{Re}[(1+\iota)\langle n|\hat{\rho}|\cos\Phi\rangle]\,, \tag{4.11}$$

since we can set

$$\langle\cos\Phi|n\rangle = \langle n|\cos\Phi\rangle\,.$$

Furthermore, using this distribution, we can represent the bosonic systems treating them as a collection of oscillators.

Now, we will suggest how the distribution may also be applied to fermionic systems.

For these systems, we can use the schematization in which a two-state system (empty-occupied), representable with the Pauli matrices [17], is associated to every level of the system.

We can consider, in order to construct the distribution, the non-commuting operators

$$\hat{\Phi}_\alpha = \hat{\eta}_\alpha(\hat{b}^*_\alpha + \hat{b}_\alpha) = (\hat{\sigma}_x)_\alpha \tag{4.12}$$

and

$$\hat{N}_\alpha = \left(\frac{1+\hat{\sigma}_z}{2}\right)_\alpha\,, \tag{4.13}$$

having $|z\alpha,\pm\rangle$, and $|x\alpha,\pm\rangle$ as eigenstates, and construct for a N-level system the distribution with the same technique used for spin-(1/2), *i.e.*

$$\hat{f}(\Phi, N) = \bigotimes_{\alpha=1}^{M} \hat{f}^{(\alpha)}(\Phi_\alpha, N_\alpha)\,. \tag{4.14}$$

5. – Comparison with the Wigner method

Before dealing with critical discussion of the distribution method, we make a comparison between the method we have just described and that of Wigner.

In the treatment we have given of the Feynman distributions, one may notice the difference in procedure compared to the Wigner method. The latter, in fact, is based on the calculation of a characteristic function, or can be obtained from group theory considerations, given for the first time by H. Weyl [18]. It is not difficult to see that the Wigner method can be translated into the language of Feynman's method. In fact in the case of Wigner distributions, the analogous of Feynman's operators $\hat{p}(i,j)$, are the operators [3]([6])

$$\widehat{w}(p,q) = \frac{1}{\sqrt{2\pi\hbar}} \int \mathrm{d}\tau\, e^{\iota p\tau/\hbar} \left| q + \frac{\tau}{2} \right\rangle \left\langle q - \frac{\tau}{2} \right| , \tag{5.1}$$

which obey an orthonormality relation with respect to the trace which is similar to (3.4a):

$$\mathrm{tr}\{\widehat{w}(p,q) \cdot \widehat{w}(p',q')\} = \delta(q-q') \cdot \delta(p-p') , \tag{5.2}$$

the normalization factor is defined as

$$v_{qp} = \mathrm{tr}\{\widehat{w}(p,q)\} = \frac{1}{\sqrt{2\pi\hbar}} , \tag{5.3}$$

whereas the operator weight is

$$O(p,q) = \mathrm{tr}\{\widehat{w}(p,q) \cdot \widehat{O}\} v_{qp}^{-1} , \tag{5.4}$$

finally, the positivity condition of physical probability is contained in the relation

$$\mathrm{tr}\{\hat{\rho} \cdot |s\rangle\langle s|\} = \iint \mathrm{d}p\, \mathrm{d}q \mathrm{tr}\{\widehat{w}(p,q) \cdot \hat{\rho}\} \mathrm{tr}\{\widehat{w}(p,q) \cdot |s\rangle\langle s|\} , \tag{5.5}$$

where $|s\rangle$ is an eigenstate of an observable. Obviously this property is due to the explicit form of $\widehat{w}(p,q)$, which is connected with the commutation relation by Weyl's argument, and is directly associable with the uncertainty relation [5].

Finally, again using the Wigner method, it is possible to treat the EPR experiment and construct a theory similar to the local hidden-variable ones [19, 20].

The only interpretative novelty, introduced by Feynman compared to the preceding method, is the interpretation of the quantity

$$\mathrm{tr}\{\widehat{w}(p,q) \cdot |s\rangle\langle s|\} v_{qp}^{-1} \tag{5.6}$$

as the probability of being in the state s, when the condition (p,q) holds, and consequently of the quantities (5.4), as the average value of the physical quantity, corresponding to the operator $\widehat{O}$. This interpretation treats the observable as if it were able to fluctuate among the eigenstates s (in each of which the observable assumes a value equal to the eigenvalue o_s) when the condition (p,q) holds, with a probability given by (5.6). Of course the same argument is also true, *mutatis mutandis*, for (3.7b). If the variable s is continuous, (5.6) becomes a probability density and the sum on s becomes an integral.

This interpretation overcomes one of the L. Cohen's criticism [8] of the "stochastic transcription" furnished by the quasiprobability distributions. This author asserts that, given an operator $\widehat{\omega} = \omega(\widehat{O})$, function of the operator $\widehat{O}$, it is impossible to obtain a distribution that verifies the property

$$\langle \omega(\widehat{O}) \rangle = \iint \mathrm{d}p\, \mathrm{d}q\, \omega(O(p,q)) \cdot f(p,q) , \tag{5.7}$$

([6]) All integrals go from $-\infty$ to $+\infty$.

for every operator, with a consequent inconsistency of the transcription procedure of quantum mechanics in terms of the classical probability theory.

If we adopt the Feynman interpretation outlined above, (5.7) no longer represents an obstacle to the quasiprobability distribution method. In fact the mean value of $\omega(\widehat{O})$, when the condition (p, q) holds, according to Feynman is

$$\omega(p, q) = \sum_s \omega(o_s) \cdot \Pi_s(p, q)$$

(with an obvious generalization to the continuous spectrum) coinciding with the quantity (5.4), calculated for $\widehat{\omega}$. Clearly $\omega(p, q) = \omega(O(p, q))$ need not necessarily hold.

Finally we note that, compared to the Feynman method, the Wigner method has the advantage of operating with the momenta and coordinates, making the analogy with the classical phase space more evident.

The Feynman distributions, on the other hand, have the advantage of being easier to handle for some systems, for example for the spin-(1/2) system.

6. – Discussion

After this comparison with the Wigner method, we will now discuss the positive and negative aspects of using quasiprobability distributions to transcribe the results of quantum theory into the classical probability language.

It is perhaps obvious that the first important defect is that we remain tied to quantum-mechanical formalism. Instead of being able to eliminate amplitudes *ab initio*, we are constrained to use typical properties of the complex amplitude formulation, in transcribing the results of this formulation into expressions which are similar to stochastic process equations. This is the main reason for which in this case we must speak of a "transcription" of quantum theory in the language of probability theory, rather than of a real "reformulation".

In addition to this first difficulty, there is also the problem of the non-uniqueness of the distributions, in other words the "transcription" we have talked about, as we have seen, can be performed in many ways. According to some authors, for example L. D. Landau and E. M. Lifschitz [21], this non-uniqueness might be a sign of the lack of direct physical meaning of the distributions. A possible answer to this problem is that Feynman interprets the negative probabilities that appear in the virtual states in the same way as an accountant treats negative quantities that appear in calculations, subtracting the total disbursements before adding the total receipts. The accountant has many ways of performing the calculation but in the end always obtains the same positive quantities. Likewise, the probability associated with the virtual states can be constructed in several ways, provided that their sum gives a final result which correctly corresponds to that of quantum mechanics, and that the physical constraint linked to the uncertainty principle, is taken into consideration.

Finally, it is worth noting that the distributions furnish a stochastic interpretation, and for this reason the fact of attributing a physical meaning to them, would raise the problem of which causes that generate the fluctuation of the physical quantity among the values that it can assume. Clearly, in this case the difficulty in solving the problem seems to be more arduous, in comparison with the stochastic formulation with a positive probability, since probabilities may assume values also external to [0,1].

In spite of the difficulties listed, the method has the merit of overcoming some contradictions of the formulation of the probability amplitude "waves", that are also non-directly detectable, with resulting interpretative advantages. For example, the method permits reasoning in a way which is closer to classical logic (concerning this, see Feynman's example of the double-slit experiment), and furthermore solves the non-locality paradoxes connected with the use of amplitudes. Consequently, all the speculation on the nature of a hypothetical superluminal signal, or the attempts to detect "empty waves", in Feynman's scheme become meaningless.

We must remember that, in spite of some aspects which are radically at odds with classical intuition, the "amplitude waves" have been proved very useful in the development of the theory so far, showing an exceptional capability to predict the experimental reality, albeit in an indirect way. From the point of view of [22] that "... the value of a scientific theory is not gauged by the faithfulness of its representation of a given class of known empirical laws, but rather by its predictive power of discovering as yet unknown facts", the "complex amplitude" formulation can scarcely be surpassed. For these reasons, we do not claim that the quasiprobability scheme can replace the "amplitude wave" formulation. Rather our aim has been to show that the freedom of allowing probabilities to be negative can bring considerable interpretative advantages.

On the basis of the argument presented in this paper, we can therefore conclude that the quasiprobability distributions may be considered as a different way of looking at the deeply and rooted divisions existing between quantum mechanics and local realism, and furthermore, allow the various paradoxical aspects of the quantum world to be considered from a new and interesting point of view.

Appendix

We will now give the proof that if the $\hat{p}(i,j)$ are an orthonormal and complete system, then Feynman's constraint is verified.

In the spin-(1/2) case, this constraint is expressed in the following words:

"... in addition to the condition that the sum of the f's is unity there is the restriction that the sum of the squares of the four f's be less than 1/2. It equals 1/2 for a pure state".

In formulas

$$\sum_{ij} f(i,j)\cdot f(i,j) \begin{cases} = \dfrac{1}{2}, & \text{if} \quad \hat{\rho} = |\Psi\rangle\langle\Psi|\,, \\ < \dfrac{1}{2}, & \text{if} \quad \hat{\rho} = \sum_{\Psi} \wp_{\Psi} |\Psi\rangle\langle\Psi|\,. \end{cases} \tag{A.1}$$

(A.1) may be written as

$$\sum_{ij} p(i,j)\cdot p(i,j) \le 1\,, \tag{A.2}$$

being $f(i,j) = \pm\frac{1}{\sqrt{2}} p(i,j)$. The expression (A.2) is therefore immediately proved, starting by ($3.5b$):

$$\sum_{ij} p(i,j)\cdot p(i,j) = \mathrm{tr}(\hat{\rho}^2) \le 1\,. \tag{A.3}$$

In the general case Feynman says:

"The condition that all physical probabilities remain positive is that the square of $p(i,j)$ not exceed the product $p(i,i)p(j,j)$ (equality is reached for pure states)".

In the light of what we have said for the spin-(1/2), we are led to think that Feynman's words refer to the following expression:

$$\sum_{ij} p(i,j) \cdot p(i,j) \leq \sum_{ij} p(i,i) \cdot p(j,j) = 1 \tag{A.4}$$

$(p(i,i) = \mathrm{Re}[(1+\iota)\langle i|\hat{\rho}|i\rangle] = \langle i|\hat{\rho}|i\rangle)$ with the equality holding in the pure state case.

In the finite-dimensional case, we can verify (A.4) in the same way as was done for the spin-(1/2) systems. In the infinite-dimensional case, a proof can be also given which is similar to that of the finite-dimensional case, because $\hat{\rho}$ belongs to the space of Hilbert-Schmidt Hermitian operators. This shows that Feynman's constraint is verified in this case too, if the operators $\hat{p}(i,j)$ are an orthonormal and complete system in this space.

* * *

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