The gravitational effect on induced charge density for an obliquely rotating neutron star(*)

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Summary. — The effect on the induced charge density of the gravitational field of a rotating neutron star with its magnetic axis inclined with respect to the rotational axis is investigated. While gravitation increases the charge density the obliquity reduces it.

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1. – Introduction

Pulsars are modelled by regarding them as neutron stars with a very high magnetic dipole field, $B \sim 10^{12}$ G, inclined at some angle, χ, to the rotational axis. The simplest analysis of the induced surface charge density, due to the rotating magnetic field, is obtained by disregarding gravitational effects and taking the angle of obliquity, χ, to be zero [1].

Even before the discovery of pulsars, Deutsch [2] had solved the Maxwell equations for an obliquely rotating magnetic star, with a magnetic field given by a point dipole at the core, subject to the boundary conditions

$$\left\{ \begin{array}{l} B_r(R) = \dfrac{2\mu}{R^3}\cos\psi\,, \\ E_\theta(R) = -\dfrac{2\mu\omega}{cR^2}\sin\theta\cos\psi\,, \\ E_\phi(R) = 0\,, \end{array} \right. \tag{1}$$

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where (r, ψ, ν) are spherical coordinates in the reference frame at rest with respect to the star, μ is the magnetic dipole moment and c is the light speed.

The resulting components of the magnetic field (in polar coordinates) are

$$(2) \qquad \begin{cases} B_r = \dfrac{2\mu}{r^3}(\cos\chi\cos\theta + \sin\chi\sin\theta\cos\lambda)\,, \\ B_\theta = \dfrac{\mu}{r^3}(\cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda)\,, \\ B_\phi = \dfrac{\mu}{r^3}\sin\chi\sin\lambda\ , \end{cases}$$

where $\lambda = \phi - \omega t$. The resulting external electric-field components, near the neutron star surface are given by

$$(3) \qquad \begin{cases} E_r = -\dfrac{\xi}{2}\dfrac{R^2}{r^2}[\cos\chi(3\cos 2\theta + 1) + 3\sin\chi\sin 2\theta\cos\lambda]\,, \\ E_\theta = -\xi\left[\dfrac{R^2}{r^2}\cos\chi\sin 2\theta + \sin\chi\left(1 - \dfrac{R^2}{r^2}\cos 2\theta\right)\cos\lambda\right]\ , \\ E_\phi = \xi\left(1 - \dfrac{R^2}{r^2}\right)\sin\chi\cos\theta\sin\lambda\ , \end{cases}$$

where $\xi = \mu\omega/(cr^2)$.

Sengupta [3], on the other hand, incorporated the effect of the gravitational field, taken as a Schwarzschild background, for an aligned dipole rotator. The effect comes in through the conversion to the inertial frame, with the orthonormal tetrad $\lambda^\mu_{(\alpha)}$ given by

$$(4) \qquad \begin{aligned} &\lambda^t_{(t)} = \left(1 - \frac{r_\mathrm{s}}{r}\right)^{-1/2}, \quad \lambda^r_{(r)} = \left(1 - \frac{r_\mathrm{s}}{r}\right)^{1/2}, \\ &\lambda^\theta_{(\theta)} = \frac{1}{r}\ , \quad \lambda^\phi_{(\phi)} = \frac{1}{r\sin\theta}\ , \end{aligned}$$

through the local Lorentz factors contained in it (here r_s is the Schwarzschild radius). The only effect is to modify the dependence of the electromagnetic fields on the radial parameter r (in a form given in the next section). This causes an enhancement of the space and surface charge densities.

In this paper we incorporate the effect of gravitation *à la* Sengupta for Deutsch's obliquely rotating magnetic star as the lowest-order correction. Of course, there would be significant changes in the physics of the obliquely rotating magnetic star due to the rotationally induced electric field. We are not going to consider these effects in any detail, as we want to concentrate on the joint effect of gravitation and rotation only. In the next section we will present the expressions for the relevant quantities in the case of the obliquely rotating massive magnetic star and in sect. **3** discuss the results. It is found that the obliquity supresses the charge density substantially while the gravitational field enhances it. Finally, in the same section, we also give a crude qualitative discussion of the effect of a finite charge density outside the star due to the rotationally induced electric field.

2. – The oblique rotator

The key point to note is that Deutsch's introduction of obliquity of the rotator has no effect on the dependence of the electromagnetic fields on the radial parameter, but modifies the dependence on the polar angles. On the other hand, Sengupta's introduction of the effects of gravitation, as embodied in the Schwarzschild background, only effects the dependence of the electromagnetic fields on the radial parameter and not on the polar angles. As such, when both effects are incorporated we obtain, for the magnetic field in the curved background

$$\begin{cases} B_r = -\dfrac{6\mu}{r_\mathrm{s}^3} a(r)(\cos\chi\cos\theta + \sin\chi\sin\theta\cos\lambda)\,, \\ B_\theta = \dfrac{3\mu}{rr_\mathrm{s}^2} b(r)(\cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda)\,, \\ B_\phi = \dfrac{3\mu}{rr_\mathrm{s}^2} b(r)\sin\chi\sin\lambda\,, \end{cases} \tag{5}$$

where

$$\begin{aligned} a(r) &= \ln\left(1-\frac{r_\mathrm{s}}{r}\right) + \frac{r_\mathrm{s}}{r}\left(1+\frac{r_\mathrm{s}}{2r}\right)\,, \\ b(r) &= \left[\frac{2r}{r_\mathrm{s}}\ln\left(1-\frac{r_\mathrm{s}}{r}\right) + \left(1-\frac{r_\mathrm{s}}{r}\right)^{-1} + 1\right]\left(1-\frac{r_\mathrm{s}}{r}\right)^{1/2}\,. \end{aligned} \tag{6}$$

The corresponding internal electric fields (near the surface, *i.e.* $r \simeq R$) become

$$\begin{cases} E_r^\mathrm{int} = \dfrac{3\mu\omega}{cr_\mathrm{s}^2} f(r)[\cos\chi\sin^2\theta - \sin\chi\sin\theta\cos\theta\cos\lambda]\,, \\ E_\theta^\mathrm{int} = \dfrac{6\mu\omega r}{cr_\mathrm{s}^3} g(r)[\cos\chi\sin\theta\cos\theta + \sin\chi\sin^2\theta\cos\lambda]\,, \\ E_\phi^\mathrm{int} = 0\,, \end{cases} \tag{7}$$

where

$$\begin{aligned} f(r) &= \left[\left(1-\frac{r_\mathrm{s}}{r}\right)^{-1} + \frac{2r}{r_\mathrm{s}}\ln\left(1-\frac{r_\mathrm{s}}{r}\right) + 1\right]\,, \\ g(r) &= \left[\ln\left(1-\frac{r_\mathrm{s}}{r}\right) + \frac{r_\mathrm{s}}{r}\left(1+\frac{r_\mathrm{s}}{2r}\right)\right]\left(1-\frac{r_\mathrm{s}}{r}\right)^{-1/2}\,. \end{aligned} \tag{8}$$

The external electric field, in the curved background, is given by

$$\begin{cases} E_r^\mathrm{ext} = \Xi\dfrac{3R^2}{r^2} f(r)[\cos\chi(3\cos^2\theta - 1) + \dfrac{3}{2}\sin\chi\sin 2\theta\cos\lambda]\,, \\ E_\theta^\mathrm{ext} = \Xi\dfrac{3r^2}{r_\mathrm{s}} g(r)\left[\dfrac{R^2}{r^2}\cos\chi\sin 2\theta + D\ \sin\chi\cos\lambda\right]\,, \\ E_\phi^\mathrm{ext} = -\Xi\dfrac{3r^2}{r_\mathrm{s}} g(r)\left(1-\dfrac{R^2}{r^2}\right)\sin\chi\cos\theta\sin\lambda\,, \end{cases} \tag{9}$$

where $\Xi = \mu\omega/(cr_\mathrm{s}^2)$ and $D = (1 - R^2\cos 2\theta/r^2)$. As for the aligned rotator, the polar and azimuthal components of the electric field are continuous across the surface of the obliquely rotating star. Only the radial component has a discontinuity at the surface. Thus, the surface charge density is given by

$$\sigma = \frac{1}{4\pi}(E_\mathrm{r}^\mathrm{ext} - E_\mathrm{r}^\mathrm{int})\,. \tag{10}$$

Inserting the values from eqs. (7) and (9) we get

$$\sigma = -\frac{3\mu\omega}{2\pi c r_{\rm s}^2} f(R)(\cos\chi\cos^2\theta + \sin\chi\sin\theta\cos\theta\cos\lambda)\ . \tag{11}$$

It is easy to verify that putting $\chi = 0$ we recover Sengupta's result for the aligned rotator (denoted by ρ there).

For the moment, neglecting any macroscopic currents near the surface (so that $\boldsymbol{\nabla}\times\mathbf{B} = 0$) the internal electric field is associated with an interior charge density

$$\rho = \frac{1}{4\pi}\boldsymbol{\nabla}\cdot\mathbf{E}^{\rm int}\ . \tag{12}$$

Equations (7) then yield

$$\rho = \Xi\frac{3}{4\pi}\Big\{\Big[\cos\chi[\sin^2\theta\ h(r) + (3\cos^2\theta - 1)k(r)] - \\ -\frac{1}{2}\sin\chi\sin 2\theta\cos\lambda[h(r) - 3k(r)]\Big]\Big\}\ . \tag{13}$$

Here

$$\begin{aligned} h(r) &= \left[\frac{4}{r}(1-\beta)^{-1} + \frac{6}{r_{\rm s}}\ln(1-\beta) - \frac{r_{\rm s}}{r^2}(1-\beta)^{-2} + \frac{2}{r}\right]\ , \\ k(r) &= \left[\frac{2}{r_{\rm s}}\ln(1-\beta) + \frac{2}{r}(1+\beta/2)\right](1-\beta)^{-1/2}. \end{aligned} \tag{14}$$

The acceleration of a point charge (q) of mass $m_{\rm p}$, above the surface of the star, along the magnetic field lines, is given by

$$a = \frac{q}{m_{\rm p}}\frac{\mathbf{E}^{\rm ext}\cdot\mathbf{B}}{|\mathbf{B}|}\ . \tag{15}$$

The general expression for this quantity, obtained from eqs. (5) and (9) is too complicated to make sense to give it directly. Sengupta [3] has given it for the aligned rotator ($\chi = 0$). Disregarding the gravitational effect we get

$$\begin{aligned} a = -\frac{q\mu\omega}{m_{\rm p} c r^2}&\{4R^2\cos^2\chi\cos^3\theta/r^2 + \\ &+ \sin^2\chi[\cos 2\lambda(1 - R^2/r^2) - 4R^2\sin^2\theta\cos^2\lambda/r^2]\cos\theta + \\ &+ \sin\chi\cos\chi[\sin\theta + R^2(3\sin^2\theta - 13\cos^2\theta)\sin\theta/r^2]\cos\lambda\}\cdot \\ &\cdot\{\cos^2\chi(3\cos^2\theta + 1) + \sin^2\chi(3\sin^2\theta\cos^2\lambda + 1) + \\ &+ 6\sin\chi\cos\chi\sin\theta\cos\theta\cos\lambda\}^{-1/2}\ . \end{aligned} \tag{16}$$

By including the gravitational effects up to the first-order correction in $r_{\rm s}/r$ we obtain that the charge acceleration a is modified by multiplying the previous expression by the factor $(1+9\beta/4)$ and adding to the three terms in the last bracket the following expression: $2\beta[\cos^2\chi(2\cos^2\theta+1)+\sin^2\chi(2\sin^2\theta\cos^2\lambda+1)+\sin 2\chi\sin 2\theta\cos\lambda]$.

3. – Discussion of the effect of obliquity

Modelling a pulsar as a neutron star with a point dipole magnetic field with rotation incorporated by the Deutsch procedure and gravitation by a Schwarzschild background is certainly a very simplistic approach. The pulsar has a complicated structure which would provide, in fact, a much more complicated magnetic field. Further, it has been noted that a rotating magnetic dipole would show relativistic effects at high rotational speeds [4, 5]. The same features were also noted for a fast rotating neutron star with a dipole field frozen on to its surface [6]. The Deutsch model [2] does not show this relativistic feature. As such, the relativistic correction needs to be incorporated. Also, the gravitational field should, correctly speaking, be obtained by solving the coupled Einstein-Maxwell field equations for a rotating, magnetic gravitational source, rather than using the Schwarzschild background. All this would make the results more realistic, but much less intellegible. The beauty of Sengupta's approach is the simplicity of the incorporation of the gravitational effect. This carries over to the oblique rotator, as modelled by Deutsch.

The surface charge density is modified from the value given, for example, in [1]

$$\sigma_{\mathrm{st}} = -\frac{\mu\omega}{2\pi c R^2}\cos^2\theta \,, \tag{17}$$

due to the inclination through χ which modifies the "$\cos^2\theta$" as in eq. (11), and due to gravitation which replaces "R^{-2}" by "$f(R)/r_{\mathrm{s}}^2$", where $f(R)$ is given by eq. (8). In the lowest-order approximation for $f(R)$ the two values coincide. More precisely

$$f(R)/r_{\mathrm{s}}^2 = \frac{1}{3}R^2 + \frac{1}{2}\frac{r_{\mathrm{s}}}{R^3} + O(r_{\mathrm{s}}^2) \,. \tag{18}$$

As such the gravitational effect enhances the surface charge. For a 1.4 $M_\odot$ neutron star of radius 10 km this enhancement is by a factor of about 2.2. On the other hand, the obliquity reduces the surface charge. In fact, the part proportional to $\sin\chi$ in eq. (11) averages out to zero over one time period, or equivalently over all ϕ (from 0 to 2π). Thus, on average we simply get

$$\bar\sigma = \sigma_{\mathrm{st}}\cos\chi \,. \tag{19}$$

Hence, for an orthogonal rotator, *i.e.* $\chi=\pi/2$, $\bar\sigma$ vanishes. There will be positive and negative charges induced but they will average out over every ring at $\theta=$const.

The space charge density, ρ, is defined by taking the divergence of the electric field in the interior of the star (the exterior electric field has zero divergence). However, the background curvature effect, by the introduction of the local Lorentz factor, could only be calculated outside the star, since the Schwarzschild interior solution would have to be used inside. The use of eqs. (8) to obtain σ is valid on account of the continuity of the metric across the surface of the star. Their use to evaluate ρ, given by eqs. (13) and (14) does not seem to make much sense. These equations have been given for comparison with the result of Sengupta (which suffers the same problem). Moreover, the classical space charge density, on integration over all θ from 0 to π gives zero. This is not apparent for

Sengupta's space charge density, or ours. However, if we incorporate the change of sign of charge density for the contribution from the northern and southern emispheres, the net charge density is zero as required.

As far as the acceleration of a charged particle is concerned, the main effect of introducing the obliquity is to reduce the acceleration by a factor $\cos\chi$. The other terms in eq. (16), in fact, average out to zero over one time period (due to the presence of $\cos\lambda$).

Pulsars have essentially two different kinds of emission mechanisms. The emission from a standard obliquely rotating magnetic dipole is given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{B^2 R^6 \omega^4}{6c^3} \sin^2\chi \,, \tag{20}$$

which is zero for an aligned rotator and maximum for an orthogonal rotator. On the other hand, the synchrotron emission comes from the charges accelerated along the magnetic field lines around the neutron star. As one can see from eqs. (13) and (14), the main terms of the expressions for ρ and a are proportional to $\cos\chi$. We can therefore conclude that nearly aligned pulsars should be very efficient emitters in the high energy band and not efficient emitters in the radio band. The contrary should happen for nearly orthogonal rotating pulsars.

It needs to be stressed that the analysis above is physically unrealistic, because the obliquely rotating dipole will induce an electric field which leads to charges being torn off the surface of the star and being accelerated by the magnetic field. As such the charge density outside the star would be non-zero, in contradiction to the Deutsch model on which our calculations are based. Realistic incorporation of the effects would again cause loss of simplicity of the Sengupta analysis. Since our analysis is only a rough approximation, it seems adequate to take a simple physical argument to estimate the effect of the correction due to finite external charge density. We know that the action of the magnetic field would be to "oppose the rotation", giving an effect like friction. Till the effects balanced there would be a net "frictional force", which would cease only at the stage where the two are in equilibrium. Without going into a detailed analysis, it seems reasonable to expect that the rotational effect would be halved due to the finite external charge.

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