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Exact path integral treatment for step potential in relativistic two-component theory

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Summary. — The path integral formalism for relativistic two-component theory (Feshbach-Villars equation) is constructed and applied firstly to the case of the free particle as a test and secondly to the important case of the step potential. We have used the perturbation method to evaluate exactly the propagators. The energy-dependent wave functions are suitably extracted. We have also pointed out that in the case of the step potential the apparition of the pair creation is expected as in the Klein-Gordon equation.

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1. – Introduction

Up to now, it remains difficult to built a relativistic one-particle theory without ambiguity because at large velocities near the speed of light the creation and the annihilation of pairs of particles are inherent in the theory. Then, the ordinary quantum mechanics, where the total number of particles is fixed, could not describe this domain and cedes its place to the quantum field theory. In spite of this limitation, there exist many physical situations where the one-particle quantum mechanics could be a good approximation. Accordingly, Dirac equation should describe a spin-(1/2) particle and spin-0 particle would be described using Klein-Gordon equation. The first candidate, Dirac equation, has been successfully considered as a single-particle equation, whereas this character has been generally avoided for Klein-Gordon equation. The main difficulty of this equation is the presence of the time second-order derivatives. Feshbach-Villars (FV) equation [1-3] is an attempt to restore this character. Their procedure concerns mainly the reduction of this order derivatives by using a two-component wave function instead of scalar function.

Therefore, the equation would exhibit the symmetry of the charge required by relativistic principles. Furthermore, there would exist a negative energy solution which could be readily interpreted as the "antiparticle".

Our aim in this paper is the treatment of a spinless particle from this point of view using path integral framework. The above formalism has been an efficacious device in treating the non-relativistic quantum mechanics problems [4]. Its application to the relativistic problems remains very restrictive and contestable because of the use of the five parameter like the Schwinger proper time [5-8]. With FV formalism this default is avoided owing to the use of the physical time as a parameter evolution by which the formalism recovers its Schrodinger meaning. There exists few problems which may be treated using this approach, among which we can cite the free-particle case, interaction with a constant magnetic field and the Coulomb potential [9]. In this paper, we wish to add to the list the one-dimensional step potential. In our analysis, we follow exactly the same method elaborated for the scalar potential [9]. Consequently, it would be necessary to resolve an integral equation of scalar type related to step potential. To this end, we use the combinatorial method [10] suggested from standard Fredholm theory. The Green's function is then calculated and the related wave functions are readily deduced. At last, we will compute the reflection and transmission coefficients relative to this barrier and in the case where this latter is strong we are obliged, as in KG equation, to introduce a pair creation to restore the equilibrium of the coefficients balance.

In sect. **2** we give some review of FV formalism and set up a path integral approach for one dimension system. The Pauli matrices describing the symmetry of the charge are replaced by the bosonic Schwinger model and an enlarged dynamics space is then used. The free case is explicitly carried out via perturbation series method as a test. In sect. **3** the step potential is considered. The energy-dependent wave functions are deduced and the reflection and transmission coefficients are then calculated. In sect. **4** the concluding remarks are given.

2. – Canonical form equation for relativistic spin-0 particle

2·1. *Review of Feshbach-Villars equation.* – The main idea to bring Klein-Gordon equation to its Hamiltonian form is to reduce the time derivative order by defining a new two component wave function according to

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi + \dfrac{i}{m}\left(\dfrac{\partial}{\partial t} + ieV\right)\Phi \\ \Phi - \dfrac{i}{m}\left(\dfrac{\partial}{\partial t} + ieV\right)\Phi \end{pmatrix}, \tag{1}$$

where Φ must obey Klein-Gordon equation.

Then, it is easily verified that Ψ will satisfy an equation of Schrodinger type, known as FV equation [6]

$$i\frac{\partial \Psi}{\partial t} = \mathcal{H}\Psi, \tag{2}$$

where the Hamiltonian matrix is given by

$$\mathcal{H} = -\frac{1}{2m}\left(\frac{\mathrm{d}}{\mathrm{d}x} - ieA\right)^2 (\tau_3 + i\tau_2) + m\tau_3 + eV, \tag{3}$$

with (V, A) is the electromagnetic potential.

The usual Pauli matrices (τ_1, τ_2, τ_3) are here introduced as isocharge matrices describing the symmetry of charge involved in the formalism and are defined by

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

From eq. (2) a current-density vector may be obtained in terms of Ψ and Ψ^+. It is satisfies a continuity equation which outlines the conservation of the charge

$$\frac{\partial \rho}{\partial t} + \frac{\mathrm{d}J}{\mathrm{dx}} = 0, \tag{5}$$

where

$$\rho = \Psi^+ \tau_3 \Psi, \tag{6a}$$

and

$$J = \frac{e}{2im}\left[\Psi^+ \tau_3 (\tau_3 + i\tau_2)\frac{\mathrm{d}}{\mathrm{d}x}\Psi - \frac{\mathrm{d}}{\mathrm{d}x}\Psi^+ \tau_3 (\tau_3 + i\tau_2)\Psi - \frac{e}{m} A \Psi^+ \tau_3 (\tau_3 + i\tau_2)\Psi\right]. \tag{6b}$$

Let us notice that the density ρ appears as the difference of two positive definite densities as would be expected in the theory describing the particles with two signs of the charge. Accordingly, it is easy to see that through the transformation of the charge conjugation

$$\Psi \to \Psi_{\mathrm{c}} = \tau_1 \Psi^*, \tag{7}$$

ρ and J will be modified as

$$\rho \to \rho_{\mathrm{c}} = -\rho, \quad J \to J_{\mathrm{c}} = J. \tag{8}$$

If Ψ is a positive energy eigenstate of the problem with a charge$(+e)$, the Ψ_{c} would be a negative energy eigenstate of the same problem with a charge $(-e)$. It is further seen that ρ is positive for positive energy and negative for negative energy and we might interpret it respectively as the charge density for particle and "antiparticle".

The definition of the expectation values of an operator O is, as expected, modified as

$$\langle O \rangle = \int \Psi^+ \tau_3 O \Psi \mathrm{d}^3 x. \tag{9}$$

In accordance with the correspondence principle, this yields the classical form of the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle O\rangle = i\langle[\mathcal{H},O]\rangle + \left\langle \frac{\partial O}{\partial t}\right\rangle . \tag{10}$$

Furthermore, the definition of hermitivity and unitarity is also modified as

$$\Omega = \overline{\Omega}, S\overline{S} = \overline{S}S = 1, \tag{11}$$

with

$$\overline{\Omega} = \tau_3 \Omega^+ \tau_3, \tag{12}$$

where Ω^+ is the habitual adjoint operator.

In the following subsection, we focus on the explicit calculation of the free Green's function using the perturbation series method.

2·2. *Path integral for spin-*0 *particle*. – Let us consider a spin-0 particle with a mass m in one dimension, subjected to the action of an electromagnetic field. The Hamiltonian describing this system is given by eq. (3). It is suitable to convert this Hamiltonian from matrix form to the scalar boson one using the well-known Schwinger boson model [11]

$$\mathcal{H} = C^+\left(\frac{(p-eA)^2}{2m}(\tau_3 + i\tau_2) + m\tau_3\right)C + eV, \tag{13}$$

where $C = \begin{pmatrix} a \\ b \end{pmatrix}$ and $C^+ = (a^+, b^+)$.

(a, a^+) and (b, b^+) are pairs of two boson verifying the usual commutation relations

$$\left[a, a^+\right] = \left[b, b^+\right] = 1, [a, b] = \left[a^+, b\right] = 0. \tag{14}$$

For further calculations, we will need some properties of related coherent states. These are defined as eigenstates of the annihilation operators a and b, namely

$$\mid \eta\rangle = \mid \alpha, \beta\rangle, a \mid \eta\rangle = \alpha \mid \eta\rangle, b \mid \eta\rangle = \beta \mid \eta\rangle. \tag{15}$$

These states can also be generated from a bosonic vacuum state $|0,0\rangle$ using the following relation:

$$\mid \eta\rangle = \exp\left[-\frac{1}{2}\left|\eta\right|^2\right] e^{A^+\eta} \mid 0,0\rangle, \tag{16}$$

where $|\eta|^2 = |\alpha|^2 + |\beta|^2$ and $A^+\eta = a^+\alpha + b^+\beta$.

The main properties of these states are

– non-orthogonality

$$\langle\eta|\eta'\rangle = \exp\left[-\frac{|\eta|^2 + |\eta'|^2}{2} + \eta^+\eta'\right], \tag{17}$$

– resolution of unity

$$\int \frac{\mathrm{d}\eta^+\mathrm{d}\eta}{\pi^2} \mid \eta\rangle\langle\eta \mid = 1. \tag{18}$$

To describe the motion of this spin-0 particle, it is necessary to extend the habitual configuration space $\mid x\rangle$ to the enlarged $\mid x, \eta\rangle$, where x generates the exterior motion and $\eta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ describes the symmetry of the charge. The propagator related to this system is given as matrix element of evolution operator between the state $\mid x_a, \eta_a\rangle$ at $t = 0$ and the state $\mid x_b, \eta_b\rangle$ at $t = T$

$$\mathcal{K}(x_b,\eta_b,x_a,\eta_a;T) = \langle x_b, \eta_b \mid U(T) \mid x_a, \eta_a\rangle, \tag{19}$$

where

$$U(T) = \mathbb{T}_\mathrm{D} \exp\left[-i\int_0^T \mathrm{d}t\mathcal{H}(t)\right], \tag{20}$$

with $\mathbb{T}_\mathrm{D}$ stands for the Dyson time ordering symbol.

In order to obtain a path integral formulation for this propagator (19), we follow the usual procedure by dividing the time interval T into $(N + 1)$ equal parts and taking the limit $N \to \infty$. Then, at each instant t_n, we introduce the following resolution of unity:

$$\int \mathrm{d}x \frac{\mathrm{d}\eta^+\mathrm{d}\eta}{\pi^2} \mid x, \eta\rangle\langle x, \eta \mid = 1. \tag{21}$$

It is easy to see that the propagator (19) takes the following discritized form:

$$\begin{aligned}\mathcal{K}(x_b,\eta_b,x_a,\eta_a;T) = \lim_{N\to\infty} \int \prod_{n=1}^{N} \mathrm{d}x_n \frac{\mathrm{d}\eta_n^+\mathrm{d}\eta_n}{\pi^2} \prod_{n=1}^{N+1} e^{-i\lambda_n} \frac{\mathrm{d}\lambda_n}{2\pi} \cdot \\ \cdot \prod_{n=1}^{N+1} \langle x_n, \eta_n \mid e^{-i\epsilon\mathcal{H}} \mid x_{n-1}, e^{i\lambda_n}\eta_{n-1}\rangle,\end{aligned} \tag{22}$$

with $\epsilon = T/(N + 1)$ and where we have introduced a projection operator

$$\mathbf{P} = \int_0^{2\pi} e^{i\lambda(C^+C-1)} \frac{\mathrm{d}\lambda}{2\pi}, \tag{23}$$

according to the two degrees of freedom representing the charge symmetry.

The matrix element of the expression (22) may be easily calculated using the property (17) and the completeness relation in the impulsion space

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}p_n}{2\pi} \mid p_n\rangle\langle p_n \mid = 1. \tag{24}$$

Hence, the result will be

$$\begin{aligned}\langle x_n, \eta_n \mid e^{-i\epsilon\mathcal{H}} \mid x_{n-1}, e^{i\lambda_n}\eta_{n-1}\rangle = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p_n}{2\pi} \exp\Bigg[ip_n(x_n - x_{n-1}) - \\ -\frac{1}{2}(|\eta_n|^2 + |\eta_{n-1}|^2) + \eta_n^+ \left(1 - i\epsilon\left(\frac{p^2}{2m}(\tau_3 + i\tau_2) + m\tau_3\right)\right) e^{i\lambda_n}\eta_{n-1}\Bigg].\end{aligned} \tag{25}$$

Inserting eq. (25) into eq. (22), we obtain the following discretized form for the propagator:

$$\begin{aligned}\mathcal{K}(x_b,\eta_b,x_a,\eta_a;T) = \lim_{N\to\infty} \int \prod_{n=1}^{N} \mathrm{d}x_n \frac{\mathrm{d}\eta_n^+\mathrm{d}\eta_n}{\pi^2} \prod_{n=1}^{N+1} e^{-i\lambda_n} \frac{\mathrm{d}\lambda_n}{2\pi} \cdot \\ \cdot \prod_{n=1}^{N+1} \exp\left[ip_n\Delta x_n - \tfrac{1}{2}(\eta_n^+\eta_n + \eta_{n-1}^+\eta_{n-1}) + \eta_n^+(1 - i\epsilon Q(n))e^{i\lambda_n}\eta_{n-1} - i\epsilon V(x_n)\right],\end{aligned} \tag{26}$$

where $Q(n)$ is the Hamiltonian matrix

$$Q(p_n, x_n) = Q(n) = \frac{(p_n - eA(x_n))^2}{2m}(\tau_3 + i\tau_2) + m\tau_3, \tag{27}$$

and $\Delta x_n = x_n - x_{n-1}$.

In the continuous form this expression will be written as

$$\begin{aligned}\mathcal{K}_0(x_b,\eta_b,x_a,\eta_a;T) = \int \mathcal{D}x\mathcal{D}p\mathcal{D}\eta^+\mathcal{D}\eta\, \bar{\mathcal{D}}\,\lambda \cdot \\ \exp\left[i\int_0^T \mathrm{d}t \left(p\,\dot{x} + \frac{i}{2}(\eta^+\,\dot{\eta_\lambda} - \dot{\eta}_\lambda^+\,\eta) - \eta^+Q(p,x)e^{i\lambda}\eta + eV(x)\right)\right],\end{aligned} \tag{28}$$

where the notation $\dot{\eta}_\lambda \simeq \dfrac{\eta_n - e^{i\lambda_n}\eta_{n-1}}{\epsilon}, \dot{\eta}_\lambda^+ \simeq \dfrac{\eta_n e^{i\lambda_n} - \eta_{n-1}}{\epsilon}$ has been used.

Let us now focus on the exact calculation of the free propagator by putting in expression (26) $V(x) = 0$ and $A(x) = 0$. Before performing this evaluation, we notice that it is possible to apply the canonical transformation known as Foldy-Wouthuysen transformation [9]. However, we will here proceed differently by using the perturbation series method.

In the further calculations instead of the propagator we introduce the energy Green's function related to the propagator via the Fourier transformation (implicitly E replaces $E + i0$)

$$\mathcal{G}_0(x_b,\eta_b,x_a,\eta_a;E) = \int_0^{\infty} \mathrm{d}T \mathcal{K}_0(x_b,\eta_b,x_a,\eta_a;T)e^{iET}. \tag{29}$$

Let us remark that in eq. (29) the integration over the time variable goes from 0 to infinity which means that the propagation backward in time has been omitted. This could be validated by the presence of the matrix τ_3 in the definition of the expectation value of the theory.

At this stage it is convenient to introduce the quantity called the promotor of the theory and which is defined by

$$\begin{aligned} \mathcal{K}_0^E(x_b,\eta_b,x_a,\eta_a;T) &= \int \mathcal{D}x\mathcal{D}p\mathcal{D}\eta^+\mathcal{D}\eta\, \bar{\mathcal{D}}\, \lambda \cdot \\ &\cdot \exp\left[i\int_0^T dt\left(p\,\dot{x} + + \tfrac{i}{2}(\eta^+\,\dot{\eta}_\lambda - \dot{\eta}^+_\lambda\,\eta) - \eta^+Q(p)e^{i\lambda}\eta + E\right)\right], \end{aligned} \tag{30a}$$

where

$$Q(p) = \frac{p^2}{2m}(\tau_3 + i\tau_2) + m\tau_3. \tag{30b}$$

In eq. (30a) the integrations over the $\{x\}$-paths are straightforward and give the conservation of the impulsion value p. Hence, the result will reduce to

$$\mathcal{K}_0^E(x_b,\eta_b,x_a,\eta_a;T) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} e^{ip(x_b-x_a)}\mathcal{P}_0^E(\eta_b,\eta_a;T), \tag{31}$$

where $\mathcal{P}_0^E(\eta_b,\eta_a;T)$ is defined by the following bosonic path integral:

$$\begin{aligned} &\mathcal{P}_0^E(\eta_b,\eta_a;T) = \\ &= \int \mathcal{D}\eta^+\mathcal{D}\eta\, \bar{\mathcal{D}}\, \lambda \exp\left[i\int_0^T \mathrm{d}t\left(\tfrac{i}{2}(\eta^+\,\dot{\eta}_\lambda - \dot{\eta}^+_\lambda\,\eta) - \eta^+Q(p)\eta + E)\right)\right]. \end{aligned} \tag{32}$$

We shall now consider the integration over the bosonic variables which can in principle easily be carried out due to its quadratic form. This method has been investigated in previous works [12] where the perturbation series techniques have been investigated.

Here, we will only apply the result of this method. Consequently, the main result of the path integral calculations of expression (32) will give

$$\mathcal{P}_0^E(\eta_b,,\eta_a;T) = \eta_b^+ \mathcal{P}_0(p;T)\eta_a e^{iET}, \tag{33}$$

where the matrix $\mathcal{P}_0(p;T)$ is given by the series

$$\mathcal{P}_0(p;T) = e^{-i\left(\frac{p^2}{2m}+m\right)\tau_3 T} + \sum_{n=1}^{\infty}\left(\frac{p^2}{2m}\right)^n \int_0^T \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 ... \int_0^{t_{n-1}} \mathrm{d}t_n \cdot \tag{34}$$
$$\cdot e^{-i\left(\frac{p^2}{2m}+m\right)\tau_3(T-t_1)} \tau_2 e^{-i\left(\frac{p^2}{2m}+m\right)\tau_3(t_1-t_2)} ... \tau_2 e^{-i\left(\frac{p^2}{2m}+m\right)\tau_3(t_n-0)}.$$

Now, by inserting eqs. (34), (33) and (32) into eq. (29) and afterward integrating over the period T by using the convolution theorem, we will get

$$\mathcal{G}_0(x_b,\eta_b,x_a,\eta_a;T) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} \exp\left[ip(x_b - x_a)\right] \eta_b^+ \mathcal{R}_0^E(p)\eta_a, \tag{35}$$

with $\mathcal{R}_0^E(p)$ is a matrix given by the following perturbation series:

$$\mathcal{R}_0^E(p) = Q^E(p) + \sum_{n=1}^{\infty}\left(\frac{p^2}{2m}\right)^n Q^E(p)\tau_2 Q^E(p)\tau_2 ... Q^E(p)\tau_2\,, \tag{36}$$

$$Q^E(p) = \frac{i}{E - \frac{p^2}{2m} + m^2}\left[E + \left(\frac{p^2}{2m} + m\right)\tau_3\right]. \tag{37}$$

Inserting eq. (37) into eq. (36) and by a straightforward calculation, it is easy to check that

$$\mathcal{R}_0^E(p) = \frac{\boldsymbol{\tau}(E)}{E^2 - (p^2 + m^2)} - (\tau_3 + i\tau_2), \tag{38}$$

where

$$\boldsymbol{\tau}(E) = E + \frac{E^2 + m^2}{2m}\tau_3 + i\frac{E^2 - m^2}{2m}\tau_2. \tag{39}$$

Substituting expression (38) into eq. (35), we easily verify that the following result holds:

$$\mathcal{G}_0(x_b,\eta_b,x_a,\eta_a;E) = \eta_b^+ \mathbb{G}_0(x_b,x_a;E)\eta_a, \tag{40}$$

where $\mathbb{G}_0(x_b,x_a;E)$ is the matrix FV Green's function for free particle obtained by a projection on the isocharge space described by the Pauli matrices.

This latter projection may be readily done and the result is given as

$$\mathbb{G}_0(x_b,x_a;E) = g_0(x_b,x_a,;E)\boldsymbol{\tau}(E) - \frac{i}{2m}\delta(x_b - x_a)(\tau_3 + i\tau_2), \tag{41}$$

with $g_0(x_b,x_a,;E)$ denoting the standard Klein-Gordon Green's function of the free particle defined by

$$g_0(x_b,x_a,;E) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} \frac{e^{ip(x_b-x_a)}}{E^2 - (p^2+m^2)} = \tag{42}$$

$$= \begin{cases} \frac{1}{2\sqrt{E^2-m^2}} e^{i\sqrt{E^2-m^2}|x_b-x_a|} & E > m\,, \\ \frac{-i}{2\sqrt{m^2-E^2}} e^{-\sqrt{m^2-E^2}|x_b-x_a|} & -m < E < m\,, \\ \frac{-1}{2\sqrt{E^2-m^2}} e^{-i\sqrt{E^2-m^2}|x_b-x_a|} & E < -m\,. \end{cases}$$

At this stage, we notice that the Green's function of form (41) has been obtained in [9] and we also point out that this is the a general feature of Green's functions in FV formalism. Let us here recall the role played by a singular term present in eq. (41). Knowing that in the presence of the scalar potential $V(x)$, the FV Hamiltonian is modified as

$$\mathcal{H}_0 \longrightarrow \mathcal{H}_0 + eV(x). \tag{43}$$

We would interpret the irregular part $\delta(x_b - x_a)$ of expression (41) as a source which multiplies the action of the potential $V(x)$ and let appear the square of the potential $V^2(x)$ as in Klein-Gordon theory. It should be observed that without this irregular term the theory might be considered as a semi-relativistic.

We turn now to evaluation of the propagator which may be obtained by taking the inverse Fourier transformation of eq. (41). To this end, it is convienent to write the following integral form of this Green's function using the residue theorem:

$$\mathbb{G}_0(x_b,x_a,;E) = \frac{i}{2\pi} \oint_{(c)} \mathrm{d}p \frac{e^{ip(x_b-x_a)}}{E^2 - (p^2+m^2)} \boldsymbol{\tau}(E,p), \tag{44}$$

where the matrix $\boldsymbol{\tau}(E,p)$ is given as

$$\boldsymbol{\tau}\ (E,p) = \left[E + \left(\frac{p^2}{2m} + m \right) \tau_3 - i \frac{p^2}{2m} \tau_2 \right], \tag{45}$$

and the curve (c) is a half-circle in the upper half-plane.

Inverting the Fourier transform, one gets the matrix free propagator in FV formalism

$$\mathbb{K}_0(x_b,x_a;T) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} e^{ip(x_b-x_a)} \left[e^{-iET} U_p \overline{U}_p - e^{iET} V_p \overline{V}_p \right], \tag{46}$$

where $E_p = \sqrt{p^2+m^2}$. U_p and V_p are the isocharge vectors defined as

$$U_p = \frac{1}{2\sqrt{mE_p}} \begin{pmatrix} m + E_p \\ m - E_p \end{pmatrix}, \tag{47a}$$

$$V_p = \frac{1}{2\sqrt{mE_p}} \begin{pmatrix} m - E_p \\ m + E_p \end{pmatrix} . \tag{47b}$$

with $\overline{U}_p = U_p \tau_3$ and $V_p = V_p \tau_3$.

From the above spectral decomposition of the propagator (46), one may identify the energy spectrum and the corresponding wave functions,

– for positive solution $E = E_p = \sqrt{p^2 + m^2}$

$$\Psi_p^{(+)}(x) = \frac{1}{\sqrt{2\pi}} e^{ipx} U_p, \tag{48a}$$

– for negative energy $E = -E_p = -\sqrt{p^2 + m^2}$

$$\Psi_p^{(-)}(x) = \frac{1}{\sqrt{2\pi}} e^{-ipx} V_p. \tag{48b}$$

As is expected, the positive and negative solutions are related by the charge conjugation transformation (7).

In the next section, we are going to consider an important and pedagogical problem of the step potential. All the techniques which will be used in this case are analogous to those of Coulomb problem [9]. We will also employ a combinatorial method, suggested from standard Fredholm theory, to solve the integral equation of the scalar type involved in the problem. To conclude this section, we shall calculate the reflection and transmission coefficients.

3. – Exact solution for the step potential

3·1. *Method for scalar potential.* – In the general case, it is difficult to set up a direct calculation as a canonical transformation which yields an exact result for a scalar potential because of the necessity of an infinite procedure diagonalizing the FV Hamiltonian given by eq. (3). For this reason, we turn to the perturbation method.

In the case of the scalar potential the FV Hamiltonian is given as

$$\mathcal{H}_v = C^+ \left(\frac{p^2}{2m} (\tau_3 + i\tau_2) + m\tau_3 \right) C + eV(x). \tag{49}$$

The path integral for the propagator related to this Hamiltonian is

$$\mathcal{K}_v(x_b, \eta_b, x_a, \eta_a; T) = \int \mathcal{D}x \mathcal{D}p \mathcal{D}\eta^+ \mathcal{D}\eta \, \bar{\mathcal{D}} \, \lambda \cdot \tag{50}$$
$$\cdot \exp \left[i \int_0^T \mathrm{d}t \left(p\dot{x} + \frac{i}{2} (\eta^+ \, \dot{\eta}_\lambda - \dot{\eta}_\lambda^+ \, \eta) - \eta^+ \left(\frac{p^2}{2m} (\tau_3 + i\tau_2) + m\tau_3 \right) e^{i\lambda} \eta - eV(x) \right) \right] .$$

As usually done [13], we expand the exponential of $eV(x)$ as a power series in e

(51)
$$\exp\left[-i\int_0^T \mathrm{d}teV\left(x(t)\right)\right] = \\ = 1+\sum_{n=1}^{\infty}\left(-ie\right)^n\int_0^T \mathrm{d}t_1...\int_0^{t_{n-1}}\mathrm{d}t_n V\left(x(t_1)\right)...V\left(x(t_n)\right).$$

Inserting this result in eq. (50), one obtains a perturbation series of the propagator

(52)
$$\mathcal{K}_v(x_b,\eta_b,x_a,\eta_a;T) = \mathcal{K}_0(x_b,\eta_b,x_a,\eta_a;T) + \sum_{n=1}^{\infty}\left(-ie\right)^n\mathcal{K}_v^{(n)}(x_b,\eta_b,x_a,\eta_a;T),$$

where $\mathcal{K}_v^{(n)}(x_b,\eta_b,x_a,\eta_a;T)$ is expressed by the following series:

(53)
$$\mathcal{K}_v^{(n)}(x_b,\eta_b,x_a,\eta_a;T) = \\ = \int_0^T \mathrm{d}t_1...\int_0^{t_{n-1}}\mathrm{d}t_n\left(\prod_{k=1}^{n}\mathrm{d}x_k\frac{\mathrm{d}\eta_k^+\mathrm{d}\eta_k}{\pi^2}\right)\mathcal{K}_0(x_b,\eta_b,x_1,\eta_1;T-t_1)\cdot \\ \cdot V(x_1)\mathcal{K}_0(x_1,\eta_1,x_2,\eta_2;t_1-t_2)V(x_2)...V(x_n)\mathcal{K}_V(x_n,\eta_n,x_a,\eta_a;t_n-0),$$

and $\mathcal{K}_0(x_b,\eta_b,x_a,\eta_a;T)$ stands for

(54)
$$\mathcal{K}_0(x_b,\eta_b,x_a,\eta_a;T) = \eta_b^+\mathbb{K}_0(x_b,x_a;T)\eta_a,$$

with $\mathbb{K}_0(x_b,x_a;T)$ given by expression (46).

The integrations over the coherent states (η^+,η) are straightforward and give a matrix representation of eq. (52)

(55)
$$\mathbb{K}_v(x_b,x_a;T) = \mathbb{K}_0(x_b,x_a;T) + \sum_{n=1}^{\infty}\left(-ie\right)^n\mathbb{K}_v^{(n)}(x_b,x_a;T),$$

where $\mathbb{K}_v(x_b,x_a;T)$ has the same form as in eq. (46) but without (η^+,η) integration and $\mathbb{K}_0(x_b,x_a;T)$ replaces $\mathcal{K}_0(x_b,\eta_b,x_a,\eta_a;T)$.

Generally, in explicit calculation, it is suitable to introduce instead the propagator the Green's function due to the simplification occurring in the problem. For this aim, we introduce the Fourier transform of eq. (55)

(56)
$$\mathbb{G}_v(x_b,x_a;E) = \mathbb{G}_0(x_b,x_a;E) + \sum_{n=1}^{\infty}\left(-ie\right)^n\mathbb{G}_v^{(n)}(x_b,x_a;E),$$

where $\mathbb{G}_v^{(n)}(x_b, x_a; E)$ is defined by

$$\mathbb{G}_v^{(n)}(x_b, x_a; E) = \\ = \int \prod_{k=1}^{n} \mathrm{d}x_k \mathbb{G}_0(x_b, x_1; E) V(x_1) \mathbb{G}_0(x_1, x_2; E) \ldots V(x_n) \mathbb{G}_n(x_n, x_a; E), \tag{57}$$

with $\mathbb{G}_0(x_b, x_a; E)$ is given by eq. (41).

All details for handling these calculations have been given in the case of Coulomb potential [9]. It is easy to show that the solution of eq. (56) is given by

$$\mathbb{G}_v(x_b, x_a, ; E) = g_v(x_b, x_a; E) \boldsymbol{\tau}(E, x_b, x_a) - \frac{i}{2m} \delta(x_b - x_a)(\tau_3 + i\tau_2), \tag{58}$$

where $g_v(x_b, x_a; E)$ verifies the integral equation of scalar type given by

$$g_v(x_b, x_a; E) = g_0(x_b, x_a; E) + \\ + \int_{-\infty}^{+\infty} \mathrm{d}x g_v(x_b, x; E) \left(-2ieEV(x) + ie^2 V^2(x)\right) g_0(x, x_a; E), \tag{59}$$

with $g_0(x_b, x_a; E)$ given by expression (42) and

$$\boldsymbol{\tau} \; (E, x_b, x_a) = \mathcal{U}(E, x_b) \mathcal{U}^+(E, x_a) \tau_3 \,, \tag{60}$$

$$\mathcal{U}(E, x) = \frac{1}{\sqrt{2m}} \begin{pmatrix} m + E - eV(x) \\ m - E + eV(x) \end{pmatrix}. \tag{61}$$

3˙2. *Application.* – In this subsection, we are concerned by an application of this FV formalism to an elementary case of step potential. This is defined by

$$V(x) = \begin{cases} V_0 \,, & x > 0, \\ 0 \,, & x < 0. \end{cases} \tag{62}$$

Our first purpose is to find a solution to eq. (59) in which we replace $V(x)$ by expression (62), namely we write

$$g_v(x_b, x_a; E) = g_0(x_b, x_a; E) + \left(-2ieEV_0 + ie^2 V_0^2\right) \int_0^{\infty} \mathrm{d}x g_v(x_b, x; E) g_0(x, x_a; E), \tag{63}$$

where $g_0(x_b, x_a; E)$ is given by eq. (42).

The above equation is an integral equation which allows a perturbation series solution based on the iteration method. As a zero approximation to $g_v(x_b, x_a; E)$ the function

$g_0(x_b,x_a;E)$ is taken. This is substituted into the right side of eq. (63) to give the first order of the approximation. This process is then repeated an infinite times and the perturbation series is obtained

$$(64)\quad g_v(x_b,x_a;E)=g_0(x_b,x_a;E)+\sum_{n=1}^{\infty}(-2ieEV_0+ie^2V_0^2)^n\int_0^{\infty}\mathrm{d}x_1...\int_0^{\infty}\mathrm{d}x_n\cdot \cdot g_0(x_b,x_1;E)g_0(x_1,x_2;E)...g_0(x_n,x_a;E).$$

Let us remark that $g_0(x_b,x_a;E)$ can be written in the following form:

$$(65)\qquad g_0(x_b,x_a;E)=\frac{1}{\alpha}G^{|x_b-x_a|},$$

where α and G are complex numbers.

At this level, we replace the continuous sum in eq. (64) by a discrete one by considering the continuous variables x_j as discrete ones, namely, we will write [10]

$$(66)\qquad g_v(x_b,x_a;E)=\frac{1}{\alpha}\sum_{n=0}^{\infty}(z(E))^n\sum_{(x_j)}G^{|x_b-x_1|+x_1-x_2|+...+|x_n-x_a|},$$

where $z(E)$ is a certain complex number and (x_j) is the nuple of the discrete variables $\{x_1,x_2,...,x_n\}$.

It interesting to notice that it is possible to connect expression (66) to the matrix calculus and then to arrive at the following matrix representation:

$$(67)\qquad g_v(x_b,x_a;E)=\frac{1}{\alpha}\left(\frac{\mathbb{G}}{1-z(E)\mathbb{G}}\right)_{x_b,x_a},$$

where $\mathbb{G}$ is the matrix defined by the following elements:

$$(68)\qquad \mathbb{G}_{x,x'}=G^{|x-x'|},$$

with x,x' indexing the elements of the matrix.

Applying this result to our equation (66), one easily checks the following expression for the Green's function $g_v(x_b,x_a;E)$, for example for $E>m$

$$(69)\qquad g_v(x_b,x_a;E)=\begin{cases}\dfrac{e^{ir_2x_b-ir_1x_a}}{r_1+r_2}, & \text{for}\quad x_b\geq 0,\quad x_a\leq 0;\\ \dfrac{1}{2r_1}\dfrac{r_1}{r_1+r_2}\left[e^{ir_2(x_b+x_a)}\left(1-\dfrac{r_1}{r_2}\right)+e^{-ir_2(x_b-x_a)}\left(1+\dfrac{r_1}{r_2}\right)\right], & \text{for } x_b\geq x_a\geq 0;\\ \dfrac{1}{2r_1}\left[e^{ir_1|x_b-x_a|}+\dfrac{r_1-r_2}{r_1+r_2}e^{-ir_1(x_b+x_a)}\right], & \text{for } x_b,x_a\leq 0,\end{cases}$$

where $r_1 = \sqrt{E^2 - m^2}$ and $r_2 = \begin{cases} \sqrt{(E - eV_0)^2 - m^2}\,, & \text{for } E - eV_0 > 0\,, \\ -\sqrt{(E - eV_0)^2 - m^2}\,, & \text{for } E - eV_0 < 0\,. \end{cases}$

All other cases are deduced from (69) by an analytical continuation over the value of the energy.

Let us now go to the matrix representation of FV Green's function relative to the step potential by inserting this result into expression (58). Thus, the explicit form for this Green's function may be written as

$$\mathbb{G}_v(x_b, x_a; E) = \frac{e^{ir_2 x_b - ir_1 x_a}}{r_1 + r_2} \boldsymbol{\tau}\ (E, x_b, x_a) - \frac{i}{2m}\delta(x_b - x_a)(\tau_3 + i\tau_2), \tag{70}$$

for $E \geq m$ and $x_b \geq 0 \geq x_a$.

In order to extract the energy-dependent wave functions, it is convenient to write this equation in its integral form where the conditions on E, x_b and x_a become implicit. This can be done using the residue theorem. Hence, it is readily shown that

$$\mathbb{G}_v(x_b, x_a; E) = \frac{i}{2\pi} \oint_{(c)} \mathrm{d}k \mathcal{J}(k, x_b, x_a) \frac{\tau(E, E_k, x_b, x_a)}{E^2 - E_k^2}, \tag{71}$$

where the curve (c) is a half-circle in the upper half-plane and the function $\mathcal{J}(k, x_b, x_a)$ is defined by

$$\begin{aligned} \mathcal{J}(k, x_b, x_a) = {}& \theta(x_b)\theta(-x_a) \left(\frac{2k}{k + k'} e^{ik' x_b - ikx_a} \right) + \\ & + \theta(-x_b)\theta(x_a) \left(\frac{2k}{k + k'} e^{ik' x_a - ikx_b} \right) + \\ & + \theta(x_b)\theta(x_a) \left(\frac{k}{k'} e^{ik'(x_b - x_a)} - \frac{k}{k'} \frac{k - k'}{k + k'} e^{ik'(x_b + x_a)} \right) + \\ & + \theta(-x_b)\theta(-x_a) \left(e^{ik(x_b - x_a)} + \frac{k - k'}{k + k'} e^{-ik(x_b + x_a)} \right), \end{aligned} \tag{72}$$

with

$$E_k = \sqrt{k^2 + m^2} \text{ and } k' = \begin{cases} \sqrt{(E_k - eV_0)^2 - m^2} & \text{for } E_k - eV_0 > 0\,, \\ -\sqrt{(E_k - eV_0)^2 - m^2} & \text{for } E_k - eV_0 < 0 \end{cases} \tag{73}$$

and

$$\begin{aligned} & \boldsymbol{\tau}\ (E, E_k, x_b, x_a) = \\ & = \frac{1}{2m} \left(E_k^2 - 2eEV_0(\theta(x_b) + \theta(x_a)) + e^2 V_0^2 \theta(x_b)\theta(x_a) \right) (\tau_3 + i\tau_2) + \\ & + \frac{1}{2}(E - eV_0\theta(x_b))(1 + \tau_1) + \frac{1}{2}(E - eV_0\theta(x_b))(1 + \tau_1) + \frac{m}{2}(\tau_3 - i\tau_2). \end{aligned} \tag{74}$$

To obtain eq. (71) we have also used the following completeness relation:

$$\frac{1}{2\pi}\oint_c \mathrm{d}k\mathcal{J}(k,x_b,x_a)=\delta(x_b-x_a). \tag{75}$$

Taking into account the asymptotic behavior of the exponential function, it is easy to check that

$$\mathbb{G}_v(x_b,x_a,;E)=\int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}\frac{i\boldsymbol{\tau}\ (E,E_k,x_b,x_a)}{E^2-E_k^2}\mathcal{J}(k,x_b,x_a). \tag{76}$$

At this level, we introduce the inverse Fourier transform to deduce the propagator of the FV equation related to the step potential. After a direct calculation, we can easily get the following result:

$$\mathbb{K}_v(x_b,x_a;T)=\int_{-\infty}^{+\infty}\mathrm{d}k(\mathcal{M}_+(E_k)-\mathcal{M}_-(E_k))\mathcal{J}(k,x_b,x_a), \tag{77}$$

$\mathcal{M}_+(E_k)$ and $\mathcal{M}_-(E_k)$ are the matrices corresponding to charge symmetry and are given by

$$\mathcal{M}_\pm(E_k)=\frac{1}{2E_k}e^{\mp iE_kT}\boldsymbol{\tau}\ (\pm E_k,E_k,x_b,x_a)\,. \tag{78}$$

To be able to deduce the wave functions from this expression we shall separate the study over two domains.

1) The first domain is defined by $eV_0>2m$:

k' is given by eq. (73) and then could be real or imaginary. So, we have k' real for k taking the values in the interval $]-\infty,-k_1[\cup]-k_2,k_2[\cup]k_1,+\infty[$, and k' imaginary for k taking the values in the interval $]-k_1,-k_2[\cup]k_2,k_1[$.

2) The second domain is defined by $eV_0<2m$:

k' takes real or imaginary value. We have k' real for k taking the values in the interval $]-\infty,-k_1[\cup]k_1,+\infty[$, and k' is imaginary for k taking the values in the interval $]-k_1,k_1[$. k_1 and k_2 are defined as, $k_1=\sqrt{eV_0(eV_0+2m)}$ and $k_2=\sqrt{eV_0(eV_0-2m)}$.

In what follows we restrict ourselves to the first domain.

In the domain of negative values of k, one changes $k\to-k$, then k'modifies as $k'\to-k'$ for real values and $k'\to k'$ for imaginary values. Replacing all these modifications and after some straightforward calculations, one gets the following form for the propagator:

$$\mathbb{K}_v(x_b,x_a;T) = \int_0^{k_2} \frac{\mathrm{d}k}{2\pi}(\mathcal{M}_+(E_k)-\mathcal{M}_-(E_k))\mathcal{J}_1(k,x_b,x_a) + \\ + \int_{k_2}^{k_1} \frac{\mathrm{d}k}{2\pi}(\mathcal{M}_+(E_k)-\mathcal{M}_-(E_k))\mathcal{J}_2(k,x_b,x_a) + \\ + \int_{k_1}^{+\infty} \frac{\mathrm{d}k}{2\pi}(\mathcal{M}_+(E_k)-\mathcal{M}_-(E_k))\mathcal{J}_3(k,x_b,x_a)\,, \tag{79}$$

where $\mathcal{J}_1(k,x_b,x_a)$, $\mathcal{J}_2(k,x_b,x_a)$, $\mathcal{J}_3(k,x_b,x_a)$ having the same form as $\mathcal{J}(k,x_b,x_a)$ with k' taking the following values:

$$k' = \begin{cases} \sqrt{(E_k-eV_0)^2-m^2}, & \text{for } k\in[0,k_2]\ , \\ i\sqrt{m^2-(E_k-eV_0)^2}, & \text{for } k\in[k_2,k_1]\ , \\ -\sqrt{(E_k-eV_0)^2-m^2}, & \text{for } k\in[k_1,+\infty[\ . \end{cases} \tag{80}$$

In order to get the expression of the wave function, it is convenient to introduce the energy variable instead of the impulsion k via the change of variable $E_k=\sqrt{k^2+m^2}$.

Then after straightforward calculations, one deduces the explicit expression for the propagator written in its spectral decomposition. For $eV_0>2m$

$$\begin{aligned} &\mathbb{K}_v(x_b,x_a;T) = \\ &= \int_m^{eV_0-m} \mathrm{d}E_k \Big[\Big(e^{-iE_kT}\Psi^{(+)}_{\rightarrow}(x_b)\overline{\Psi}^{(+)}_{\rightarrow}(x_a)+e^{iE_kT}\Psi^{(-)}_{\leftarrow}(x_b)\overline{\Psi}^{(-)}_{\leftarrow}(x_a)\Big) - \\ &-\Big(e^{+iE_kT}\Psi^{(-)}_{\rightarrow}(x_b)\overline{\Psi}^{(-)}_{\rightarrow}(x_a)+e^{-iE_kT}\Psi^{(+)}_{\leftarrow}(x_b)\overline{\Psi}^{(+)}_{\leftarrow}(x_a)\Big)\Big] + \\ &+\int_{eV_0-m}^{eV_0+m} \mathrm{d}E_k \Big[e^{-iE_kT}\Psi^{(+)}_{\rightarrow}(x_b)\overline{\Psi}^{(+)}_{\rightarrow}(x_a)-e^{iE_kT}\Psi^{(-)}_{\rightarrow}(x_b)\overline{\Psi}^{(-)}_{\rightarrow}(x_a)\Big] + \\ &+\int_{eV_0+m}^{+\infty} \mathrm{d}E_k \Big[e^{-iE_kT}\Big(\Psi^{(+)}_{\rightarrow}(x_b)\overline{\Psi}^{(+)}_{\rightarrow}(x_a)+\Psi^{(+)}_{\leftarrow}(x_b)\overline{\Psi}^{(+)}_{\leftarrow}(x_a)\Big) - \\ &-e^{iE_kT}\Big(\Psi^{(+)}_{\rightarrow}(x_b)\overline{\Psi}^{(+)}_{\rightarrow}(x_a)+\Psi^{(+)}_{\leftarrow}(x_b)\overline{\Psi}^{(+)}_{\leftarrow}(x_a)\Big)\Big]\ , \end{aligned} \tag{81}$$

where we have

1) for $m<E_k<eV_0-m$

– the left-propagating positive energy

$$\Psi^{(+)}_{\rightarrow}(x) = \frac{1}{\sqrt{8\pi mk}}\left[\theta(-x)(e^{ikx}+R_-e^{-ikx})+\theta(x)\sqrt{\frac{k}{k'}}T_-e^{-ik'x}\right]U_k(x), \tag{82a}$$

– the right-propagating positive energy

$$\Psi^{(+)}_{\leftarrow}(x) = \frac{1}{\sqrt{8\pi mk}}\left[\theta(x)\sqrt{\frac{k}{k'}}(e^{ik'x} - R_- e^{-ik'x}) - \theta(-x)T_- e^{-ikx}\right]U_k(x), \tag{82b}$$

and

– the left-propagating negative energy

$$\Psi^{(-)}_{\rightarrow}(x) = \frac{1}{\sqrt{8\pi mk}}\left[\theta(-x)(e^{-ikx} + R^*_- e^{ikx}) + \theta(x)\sqrt{\frac{k}{k'}}T^*_- e^{ik'x}\right]V_k(x)\,, \tag{82c}$$

– the right-propagating negative energy

$$\Psi^{(-)}_{\leftarrow}(x) = \frac{1}{\sqrt{8\pi mk}}\left[\theta(x)\sqrt{\frac{k}{k'}}(e^{-ik'x} - R^*_- e^{ik'x}) - \theta(-x)T^*_- e^{ikx}\right]V_k(x) \tag{82d}$$

with $R_- = \frac{k+k'}{k-k'}$, $T_- = \frac{2\sqrt{kk'}}{k-k'}$, $k = \sqrt{E_k^2 - m^2}$ and $k' = \sqrt{(E_k - eV_0)^2 - m^2}$.

2) For $eV_0 - m < E_k < eV_0 + m$

– the left-propagating positive energy

$$\Psi^{(+)}_{\rightarrow}(x) = \frac{1}{\sqrt{8\pi mk}}\left[\theta(-x)(e^{ikx} + Re^{-ikx}) + \theta(x)\sqrt{\frac{k}{ik''}}Te^{-k''x}\right]U_k(x), \tag{83a}$$

– the left-propagating negative energy

$$\Psi^{(-)}_{\rightarrow}(x) = \frac{1}{\sqrt{8\pi mk}}\left[\theta(-x)(e^{-ikx} + R^* e^{ikx}) + \theta(x)\sqrt{\frac{k}{-ik''}}T^* e^{-k''x}\right]V_k(x) \tag{83b}$$

with $R = \frac{k-ik''}{k+ik''}$, $T = \dfrac{2\sqrt{ikk''}}{k+ik''}$ and $k'' = \sqrt{m^2 - (E_k - eV_0)^2}$.

3) For $eV_0 + m < E_k < \infty$

– the left-propagating positive energy

$$\Psi^{(+)}_{\rightarrow}(x) = \frac{1}{\sqrt{8\pi mk}}\left[\theta(-x)(e^{ikx} + R_+ e^{-ikx}) + \theta(x)\sqrt{\frac{k}{k'}}T_+ e^{ik'x}\right]U_k(x), \tag{84a}$$

– the right-propagating positive energy

$$\Psi^{(+)}_{\leftarrow}(x) = \frac{1}{\sqrt{8\pi mk}}\left[\theta(x)\sqrt{\frac{k}{k'}}(e^{ik'x} - R_+ e^{-ik'x}) + \theta(-x)T_+ e^{-ikx}\right]U_k(x), \tag{84b}$$

and

–the left-propagating negative energy

$$\Psi^{(-)}_{\rightarrow}(x) = \frac{1}{\sqrt{8\pi mk}}\left[\theta(-x)(e^{-ikx} + R^*_+ e^{ikx}) + \theta(x)\sqrt{\frac{k}{k'}}T^*_+ e^{-ik'x}\right]V_k(x)\,, \tag{84c}$$

– the right-propagating negative energy

$$\Psi^{(-)}_{\leftarrow}(x) = \frac{1}{\sqrt{8\pi mk}}\left[\theta(x)\sqrt{\frac{k}{k'}}(e^{-ik'x} - R^*_+ e^{ik'x}) + \theta(-x)T^*_+ e^{ikx}\right]V_k(x) \tag{84d}$$

with $R_+ = \frac{k-k'}{k+k'}$, $T_+ = \frac{2\sqrt{kk'}}{k+k'}$ and in all above formulas $U_k(x)$ and $V_k(x)$ are defined by

$$U_k(x) = \frac{1}{\sqrt{2m}}\begin{pmatrix} m + E - eV_0\theta(x) \\ m - E + eV_0\theta(x)\end{pmatrix}, \tag{85a}$$

$$V_k(x) = \frac{1}{\sqrt{2m}}\begin{pmatrix} m - E + eV_0\theta(x) \\ m + E - eV_0\theta(x)\end{pmatrix}. \tag{85b}$$

For the case $eV_0 < 2m$ the result can be readily deduced following the same steps of the above method but because of the longer we do not give the result here.

Now, we are ready to evaluate the related reflection and transmission coefficients and to show that in the two component theory the Klein paradox persists. This fact demonstrates the equivalence between the KG equation and FV equation. It is easy to show that we have, as in KG theory, the following charge current balance for the positive-energy wave functions:

$$\mathbb{R}^{(+)} + \mathbb{T}^{(+)} = 1 \quad \text{for} \quad eV_0 + m < E_k < \infty\,, \tag{86a}$$

$$\mathbb{R}^{(+)} = 1 \quad \text{and} \quad \mathbb{T}^{(+)} = 0 \quad \text{for} \quad eV_0 - m < E_k < eV_0 + m\,, \tag{86b}$$

$$\mathbb{R}^{(+)} - \mathbb{T}^{(+)} = 1 \quad \text{for} \quad m < E_k < eV_0 - m\,, \tag{86c}$$

where $(\mathbb{R}^{(+)}, \mathbb{T}^{(+)})$ are the reflection and transmission coefficients related to the positive energy given by

$$\mathbb{R}^{(+)} = \frac{\left| J^{(+)}_{\rightleftarrows}, \text{ref} \right|}{\left| J^{(+)}_{\rightleftarrows}, \text{inc} \right|} = \frac{2\sqrt{kk'}}{k \pm k'}, \mathbb{T}^{(+)} = \frac{\left| J^{(+)}_{\rightleftarrows}, \text{tr} \right|}{\left| J^{(+)}_{\rightleftarrows}, \text{inc} \right|} = \frac{k \mp k'}{k \pm k'}\,, \tag{87}$$

where the sign $\pm$ stands for the different regions and

$$J^{(+)}_{\rightleftarrows,.}(x) = \frac{e}{2im}\left[\overline{\Psi}^{(+)}_{\rightleftarrows,.}(\tau_3 + i\tau_2)\frac{\mathrm{d}}{\mathrm{d}x}\Psi^{(+)}_{\rightleftarrows,.} - \frac{\mathrm{d}}{\mathrm{d}x}\overline{\Psi}^{(+)}_{\rightleftarrows,.}(\tau_3 + i\tau_2)\Psi^{(+)}_{\rightleftarrows,.}\right], \tag{88}$$

the dot (.) stands for incident, reflected and transmitted wave functions.

These expressions calculated in Feshbach-Villars formalism coincide exactly with those of KG equation. In the domain $m < E_k < eV_0 - m$, we point out that instead of the sign $(+)$ we have the sign $(-)$ between the reflected and trasmitted coefficients which implies that $\mathbb{R}^{(+)} > 1$. This anomaly is restored by the introduction of the pair creation. This latter do not surprise due to the equivalence connecting the two formalisms.

4. – Conclusion

In this paper we have constructed a path integral approach for FV equation and we have treated the free case and a step potential. Using the perturbation method, we have calculated explicitly, once again, the free Green's function and the result agrees with the one deduced by canonical transformation. In the case of the step potential we have followed the same method elaborated for the Coulomb potential. In view of difficulties occurring in the problem, we believe that it should be possible to avoid perturbation method by using, for example, Duru-Kleinert transformation which could extend the list of solvable potential. In each treated case we have set up the relative propagator via the Green's function and then have deduced the corresponding wave functions. We have firstly shown that the reflection and the transmission coefficients relative to the barrier coincide exactly with those of Klein-Gordon equation and that in the case of the strong potential the apparition of the pair creation is necessary to restore the balance equilibrium.

Finally, let us signal that the same problem in the case of Feshbach-Villars equation for spin 1/2 will be considered elsewhere.

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