IL NUOVO CIMENTO VOL. 112 B, N. 10 Ottobre 1997

A local symmetry associated with the little groups of the four-velocity and the Pauli-Lubanski spin vector

P. L. NASH(*)

Division of Earth and Physical Sciences, The University of Texas at San Antonio San Antonio, Texas 78249-0663, USA

(ricevuto il 14 Febbraio 1997; approvato il 29 Aprile 1997)

Summary. — It is shown that the (quantum or classical) observables for a spin-$\frac{1}{2}$ massive electron are invariant under a local one-parameter group of gauge transformations whose generator is dependent on the fermion "fields" of the theory, as well as conventional (constant) Lie algebra generators. This one-parameter family of local transformations is $2 \to 1$ homomorphic to a set of operators common to both the "little group" of the four-velocity and the "little group" of the Pauli-Lubanski spin vector. This symmetry is called the "local little symmetry" to emphasize its connection to Wigner's "little group". The physical origin of this "local little symmetry" is discussed. This symmetry is employed to explain a degeneracy exhibited by some *classical* models of an electron with spin.

PACS 03.65 – Quantum mechanics.
PACS 03.20 – Classical mechanics of discrete systems: general mathematical aspects.
PACS 03.30 – Special relativity.

1. – Introduction

In the Dirac theory of the electron a complex four-component Dirac spinor ϕ that satisfies the Dirac wave equation may be used to define an orthogonal tetrad $E^{\alpha}_{(\mu)}$ in Minkowski space-time M_4. The timelike member $E^{\alpha}_{(4)}$ of the tetrad is the four-velocity of the particle described by ϕ, and is the sum of two future-pointing null (lightlike) vectors that are defined in terms of the components of ϕ. The difference of these two null vectors is the spacelike Pauli-Lubanski spin vector $E^{\alpha}_{(3)}$ (see below for a brief overview). The remaining two spacelike members $E^{\alpha}_{(1)}$ and $E^{\alpha}_{(2)}$ of the tetrad may also be defined in terms of ϕ, and the plane $E^{\alpha}_{(1)}E^{\beta}_{(2)} - E^{\beta}_{(1)}E^{\alpha}_{(2)}$ they span defines an anti-symmetric spin tensor $\Sigma_{\alpha\beta}$. $E^{\alpha}_{(1)}$ and $E^{\alpha}_{(2)}$ do not directly couple to external fields in a way that affects the electron's

(*) E-mail: nsh@susan.ep.utsa.edu

energy, free-field momentum or spin, and it is evidently impossible to measure their absolute orientation. This is the origin of the symmetry that is discussed in this note.

Let $S^{\alpha\beta}$ denote the generators of $\overline{SO(3,1)}$; we shall show that $\frac{1}{2}\ \Sigma_{\alpha\beta}\ S^{\alpha\beta}$ is the generator of a one-parameter group $e^{\frac{1}{2}\theta\ \Sigma_{\alpha\beta}\ S^{\alpha\beta}}$ of local gauge transformations that preserve both the four-velocity and the Pauli-Lubanski spin vector. (This transformation is local because the generator is local, regardless of whether or not the group parameter θ is local.) Since external electromagnetic fields couple only to the four-velocity and the Pauli-Lubanski spin vector (in an instantaneous rest frame), the energy of the particle is invariant under this symmetry.

Classical models of massive particles with spin that employ spinors to carry the spin degrees of freedom and which respect the Frenkel condition $\Sigma_{\alpha\beta}\dot{x}^{\beta} = 0$ possess a doubling of certain spinor states. The physical basis of this degeneracy will be explained in terms of local little symmetry.

2. – Conventions and geometrical preliminaries

Units are used in which the speed of light is one. The Minkowski space-time metric is $\eta_{\alpha\beta} = \mathrm{diag}(1,1,1,-1)$, where Greek indices run from one to four (four is used instead of zero in order to comply with Dirac's 1963 notation [1], which is adopted in this paper). Following Dirac, tilde denotes transpose.

Dirac [1] has given a real 4×4 irreducible representation of $\overline{SO(3,3)}$ that we find very useful for calculations. D_4 denotes the real four-dimensional vector space that carries this irreducible representation, and D_4^* denotes the vector space that is dual to D_4.

The elements $\lambda \in D_4$ are called (real) contravariant spinors, and have (real) spinor components denoted by λ^a, where $a, b, \ldots = 1, \ldots, 4$. An element $\xi \in D_4^*$ is called a (real) covariant spinor, and has components denoted by ξ_a. $\xi_a \lambda^a = \xi\lambda = \mathrm{tr}(\lambda\xi)$ and $\xi\gamma^5\lambda = \mathrm{tr}(\gamma^5\lambda\xi)$ are invariant under $\overline{SO(3,1)}$, and reflect the standard matrix notation used in this paper.

The symplectic form ϵ on D_4 is invariant under $\overline{SO(3,1)}$: if $S \in \overline{SO(3,1)}$ then $\tilde{S}\epsilon S = \epsilon$ under the action of $\overline{SO(3,1)}$. ϵ (respectively ϵ^{-1}) is used to lower (respectively, raise) spinor indices according to

$$\tilde{\lambda} \leftrightarrow \lambda_b = \lambda^a \epsilon_{ab} \tag{1}$$

and

$$-\epsilon^{-1}\tilde{\xi} \leftrightarrow \xi^a = \epsilon^{ab}\xi_b. \tag{2}$$

This implies that $\epsilon^{ab}\epsilon_{bc} = -\delta^a_c$.

We note that an arbitrary complex four-component Dirac spinor ϕ can be expressed as $\phi = \lambda - i\gamma^5\epsilon^{-1}\tilde{\xi}$, which may be used to define λ and ξ given ϕ. For simplicity, we shall calculate primarily with λ and ξ and not explicitly with ϕ.

Dirac [1] has labeled the 15 (real, 4×4, linearly independent, traceless) generators of his irreducible representation of $\overline{SO(3,3)}$ as $-\frac{1}{2}\gamma^{AB}$, where $A, B, \ldots = 1, \ldots, 6$. In the present paper, Dirac's gamma matrices are $\gamma^\alpha = \gamma^{\alpha 6}$ and $\gamma^5 = \gamma^{56}$.

Before constructing the tetrad from ϕ we make the following remark. The only $\overline{SO(3,1)}$ scalars that can be formed from an arbitrary contravariant spinor $\lambda \in D_4$ are functions of $\tilde{\lambda}\epsilon\lambda$ and $\tilde{\lambda}\epsilon\gamma^5\lambda$. However, both of these scalars vanish because of the skew symmetry of ϵ

and $\epsilon\gamma^5$. Therefore

$$n^\alpha = \tilde{\lambda}\epsilon\gamma^\alpha\lambda \tag{3}$$

must satisfy $n^\alpha n_\alpha = 0$ and hence must be a null (lightlike) vector. ϵ is chosen so that $\epsilon\gamma^4 = \gamma_0 \equiv$ unit matrix, which makes n^α future-pointing.

Similarly, given an arbitrary covariant spinor $\xi \in D_4^*$ one can construct the future-pointing null vector

$$m^\alpha = -\xi\gamma^\alpha\epsilon^{-1}\tilde{\xi}. \tag{4}$$

Using the identity [2]

$$\gamma_\alpha\lambda\xi\gamma^\alpha = \gamma_0\xi\lambda + \gamma^5\xi\gamma^5\lambda + \epsilon^{-1}\tilde{\xi}\tilde{\lambda}\epsilon + \gamma^5\epsilon^{-1}\tilde{\xi}\tilde{\lambda}\epsilon\gamma^5\,, \tag{5}$$

one may show that

$$m^\alpha n_\alpha = -8(N_0^2 + N_5^2), \tag{6}$$

where $N_0 = \frac{1}{2}\xi\lambda$ and $N_5 = -\frac{1}{2}\xi\gamma^5\lambda$ are $\overline{SO(3,1)}$ scalars. If $N_0^2 + N_5^2 \neq 0$, then m^α and n^α are linearly independent. In this case

$$E^\alpha_{(4)} = \frac{1}{4}(m^\alpha + n^\alpha) \tag{7}$$

is timelike and parallel to the four-velocity when ϕ satisfies the free-field Dirac equation. Also

$$E^\alpha_{(3)} = \frac{1}{4}(m^\alpha - n^\alpha) \tag{8}$$

is spacelike and corresponds to the Pauli-Lubanski spin vector. Since the Lorentz-invariant scalar products of n^α and m^α with the spin vector are $\pm\frac{1}{4}n_\alpha m^\alpha$, m^α and n^α have opposite helicities.

For completeness we note that $E^\alpha_{(1)} = \frac{1}{2}\xi\gamma^\alpha\gamma^5\lambda$ and $E^\alpha_{(2)} = \frac{1}{2}\xi\gamma^\alpha\lambda$. The tetrad verifies $\eta_{\alpha\beta}E^\alpha_{(\mu)}E^\beta_{(\nu)} = \eta_{(\mu)(\nu)}(N_0^2 + N_5^2)$ [3].

The generators of $\overline{SO(3,1)}$ are

$$S^{\alpha\beta} = -\frac{1}{4}[\gamma^\alpha, \gamma^\beta] = -\frac{1}{2}\gamma^{\alpha\beta}. \tag{9}$$

We define the spin tensor as

$$\Sigma^{\alpha\beta} = \xi S^{\alpha\beta}\lambda. \tag{10}$$

It has been shown [3] that $\Sigma^\alpha{}_\beta n^\beta = N_0\, n^\alpha$ and $\Sigma^\alpha{}_\beta m^\beta = -N_0\, m^\alpha$, so that when $N_0 = 0$

$$N_5\,\Sigma^{\alpha\beta} = E^\alpha_{(1)}E^\beta_{(2)} - E^\beta_{(1)}E^\alpha_{(2)}, \tag{11}$$

$$\Sigma^\alpha{}_\beta E^\beta_{(4)} = 0 \tag{12}$$

and

(13) $$\Sigma^{\alpha}{}_{\beta}E^{\beta}_{(3)} = 0.$$

3. – Local little symmetry

In both the Dirac wave theory of the electron interacting with an external electromagnetic field $F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$, and acceptable classical models of the interacting spin-$\frac{1}{2}$ electron, the electron's energy does not depend on the absolute orientation of $E^{\alpha}_{(1)}$ and $E^{\alpha}_{(2)}$ in the 2-plane orthogonal to the plane spanned by the four-velocity and Pauli-Lubanski spin vector. $E^{\alpha}_{(1)}$ and $E^{\alpha}_{(2)}$ do not directly couple to external fields the way the four-velocity and spin vector do. The four-velocity couples with a term proportional to $A_{\alpha}E^{\alpha}_{(4)}$ in quantum mechanics and $A_{\alpha}\dot{x}^{\alpha}$ classically. Classically, the spin vector's interaction with the field in the Lagrangian is through $\frac{1}{2}F_{\alpha\beta}\Sigma^{\alpha\beta}$, which is an interaction of $E^{\alpha}_{(3)}$ with the external magnetic field as seen in an instantaneous rest frame of the particle. This follows from the standard definition of the Pauli-Lubanski spin vector $E_{\alpha\ (3)} \propto \epsilon_{\alpha\beta\mu\nu}\Sigma^{\beta\mu}\dot{x}^{\nu}$ and the Frenkel condition $\Sigma_{\alpha\beta}\dot{x}^{\beta} = 0$.

$\Sigma_{(1)(2)} = \frac{1}{2}\ \Sigma_{\alpha\beta}\ S^{\alpha\beta} \propto S_{\alpha\beta}E^{\alpha}_{(1)}E^{\beta}_{(2)}$ is the generator of a one-parameter group $S_{\mathrm{L}}(\theta) = e^{\frac{1}{2}\theta\ \Sigma_{\alpha\beta}\ S^{\alpha\beta}}$ of local gauge transformations that preserves both the four-velocity and the Pauli-Lubanski spin vector. This can be seen by considering this action on vectors. This local $\overline{SO(3,1)}$ transformation induces a local $SO(3,1)$ transformation acting on space-time vectors defined by the well-known canonical 2-1 homomorphism $\overline{SO(3,1)} \rightarrow SO(3,1)$. Let $S \in \overline{SO(3,1)}$ and $L \in SO(3,1)$; then $S^{-1}\gamma^{\alpha}S = L^{\alpha}{}_{\beta}\gamma^{\beta}$. An infinitesimal gauge transformation $S_{\mathrm{L}}(\delta\theta) = \gamma_0 + \frac{1}{2}\ \delta\theta\ \Sigma_{\alpha\beta}\ S^{\alpha\beta}$ induces the transformation $V^{\alpha} \mapsto (\delta^{\alpha}_{\beta} + \delta\theta\ \Sigma^{\alpha}{}_{\beta})V^{\beta}$ of a space-time vector V^{α}. By virtue of eq. (12) and eq. (13) one sees that $E^{\alpha}_{(4)}$ and $E^{\alpha}_{(3)}$ are invariant under this transformation. The one-parameter family of local transformations is therefore $2 \rightarrow 1$ homomorphic to the intersection of the "little group" of the four-velocity and the "little group" of the Pauli-Lubanski spin vector. We accordingly call the symmetry the "local little symmetry" to emphasize its connection to Wigner's "little group."

The gauge transformation is local because the generator $\frac{1}{2}\ \Sigma_{\alpha\beta}\ S^{\alpha\beta}$ is local ($\Sigma_{\alpha\beta}$ is local), regardless of whether or not the group parameter θ is local.

4. – Classical dynamics

Some insight into this symmetry can be gained from the study of the classical theory of a massive point particle with spin.

The Lorentz force equations for the four-velocity $\dot{x}^{\alpha} \stackrel{\text{def}}{=} \mathrm{d}x^{\alpha}/\mathrm{d}\tau$ coupled with the Thomas-Frenkel-Bargmann-Michael-Telegdi (TFBMT) equations for the dynamical evolution of the Pauli-Lubanski spin vector govern the motion of a polarized particle in the limit $\hbar \rightarrow 0$ [4, 5]. Here τ is the proper time. A generalization of these accepted equations of motion may be *derived from a Lagrangian* when the classical dynamical variables are chosen to be the particle's position x^{α} in space-time, and real spinors λ and ξ that carry the spin degrees of freedom[3]. (Of course, the classical dynamical Lorentz and TFBMT equations have been studied and generalized by many investigators[3].) We emphasize that in the limit $\hbar \to 0$ the Lagrangian approach of [3] yields both the Lorentz force equation and the TFBMT equation exactly.

The classical dynamical variables $\{x^\alpha, \lambda, \xi\}$ are not all independent. This is because the timelike $E^\alpha_{(4)}$ constructed from arbitrary spinors $\{\lambda, \xi\}$ is in general not parallel to $\dot{x}^\alpha$. In the free-field case one knows that only one timelike vector is required for the description of massive particle 'dynamics.' In order for $\{x^\alpha, \lambda, \xi\}$ to describe a massive particle with spin the spinors must be constrained so that $E^\alpha_{(4)}$ and $\dot{x}^\alpha$ are parallel. The condition that enforces this constraint resembles the Dirac equation. To show this, we write $\{\lambda, \xi\}$ as a real eight-component spinor ψ. ψ may be regarded as a real column vector comprised of the direct sum of a real contravariant four-component Dirac spinor λ (transforming under the *real* irreducible representation of $\overline{SO(3,3)}$ generated by $S^{\alpha\beta}$) and the transpose of a real covariant four-component spinor ξ (that transforms under the irreducible representation generated by $-\tilde{S}^{\alpha\beta}$)

$$\psi = \begin{pmatrix} \lambda \\ \tilde{\xi} \end{pmatrix}. \tag{14}$$

It has been shown [6] that ψ defines an element of the (split) octonion algebra.

The timelike member $E^\beta_{(4)}$ of the tetrad can be permanently aligned with the four-velocity $\dot{x}^\alpha$ of the particle, so that $E^\beta_{(4)} \propto \dot{x}^\beta$, by imposing the constraint

$$\left(\Gamma_\alpha \dot{x}^\alpha + \sqrt{-\dot{x}_\alpha \dot{x}^\alpha}\, \Gamma^7\right) \psi = 0, \tag{15}$$

where the Γ_α, Γ^7 matrices are real 8×8 anti-commuting matrices that are direct analogs of Dirac's 4×4 gamma matrices [3]. This *classical* constraint bears a striking resemblance to the quantum-mechanical Dirac equation, but one should keep in mind that this is a purely classical model. It has also been shown that eq. (15) guarantees that $N_0 = 0$ and $N_5 \neq 0$ [3]. The constraint of eq. (15) reduces the real spinor degrees of freedom from eight to four. As noted previously [7], there is still a doubling of the spinor spin-up and spin-down states (but not a doubling of the Pauli-Lubanski spin vector spin-up and spin-down states). The physical basis of this degeneracy can be explained in terms of local little symmetry as follows.

Since constraint equation (15) implies that $N_0 = 0$, eq. (12) holds and one sees that this classical model predicts that the intrinsic electric-dipole moment of the point particle vanishes in a rest frame $\Sigma_{\alpha\beta} E^\beta_{(4)} = 0 = \Sigma_{\alpha\beta} \dot{x}^\beta$, as required by experiment. Further, by eq. (13), $\Sigma_{\alpha\beta} E^\beta_{(3)} = 0$. Therefore, this classical model possesses "local little symmetry" that preserves both the spin vector and four-velocity. Using $\left(\Gamma^7\right)^2 = I$ and $-\dot{x}_\alpha \dot{x}^\alpha = 1$ we rewrite eq. (15) as $\Gamma_\alpha \Gamma^7 \dot{x}^\alpha \psi = \psi$. Given an arbitrary normalized future-pointing (positive energy) timelike $\dot{x}^\alpha$, we solve for the four eigenvectors ψ of $\Gamma_\alpha \Gamma^7 \dot{x}^\alpha$ whose eigenvalues are $+1$ that satisfy this equation, and calculate the corresponding Pauli-Lubanski spin vectors and spin tensors $\Sigma_{\alpha\beta}$ for each. The calculation is straightforward and is omitted. We find that there is a doubling of the spinor spin-up and spin-down states, and that both spin-up spinors yield the same spin-up Pauli-Lubanski spin vector $E^\alpha_{(3)}$ and spin tensor $\Sigma_{\alpha\beta}$, and both spin-down spinors yield the same spin-down Pauli-Lubanski spin vector and spin tensor. We find by direct calculation that the local-little-symmetry transformation $S_{\mathrm{L}}(-\pi) = e^{-\frac{1}{2}\pi\, \Sigma_{\alpha\beta}\, S^{\alpha\beta}}$ maps the doubled spinor spin-up states into each other modulo ± 1, and similarly for the two spinor spin-down states. We should emphasize that the doubling of states under local little symmetry is distinct from the fact that $\pm\psi$ both map to the same Pauli-Lubanski spin vector.

In order to gain further insight into this symmetry, let us next consider the problem of gauging the local little symmetry in quantum mechanics by introducing a gauge field $G_{\alpha}^{(1)(2)}$ associated with the local little symmetry generator $\Sigma_{(1)(2)} = \frac{1}{2}\,\Sigma_{\alpha\beta}\,S^{\alpha\beta}$.

5. – Quantum dynamics

The free-field Dirac equation may be realized as ($\hbar = 1$)

$$(i\gamma^{\alpha} p_{\alpha} + m)\phi = 0. \tag{16}$$

Using the relation $\phi = \lambda - i\gamma^5\epsilon^{-1}\tilde{\xi}$, and the standard definition $\bar{\phi} = \phi^{\dagger}\gamma^4 = -\phi^{\dagger}\epsilon = -\tilde{\lambda}\epsilon + i\xi\gamma^5$, one finds that $N_5 = \frac{i}{4}\bar{\phi}\phi$ and $N_0 = \frac{i}{4}\bar{\phi}\gamma^5\phi$. For plane-wave solutions to the free-field Dirac equation, $m\bar{\phi}\phi = -i\bar{\phi}\gamma^{\alpha}p_{\alpha}\phi \neq 0$ and $m\bar{\phi}\gamma^5\phi = -i\bar{\phi}\gamma^5\gamma^{\alpha}\phi\; p_{\alpha} = 0$, since $\bar{\phi}\gamma^{\alpha}\phi = -4E^{\alpha}_{(4)}$, $i\bar{\phi}\gamma^5\gamma^{\alpha}\phi = 4E^{\alpha}_{(2)}$ and $p_{\alpha} \propto E^{\alpha}_{(4)}$. One sees that $N_0 = 0$ and $N_5 \neq 0$, so that the free-field observables are invariant under the local little symmetry.

As it stands, the Lagrangian that generates the Euler-Lagrange equation (16) is not invariant under local little symmetry. In order to construct a gauge-invariant Lagrangian, one usually generalizes the gradient ∂_{α} to the gauge-covariant derivative. The naive procedure is to introduce a gauge field $G_{\alpha}^{(1)(2)}$ associated with the local little symmetry generator $\Sigma_{(1)(2)} = \frac{1}{2}\;\Sigma_{\alpha\beta}\;S^{\alpha\beta}$ and adopt the minimal-coupling postulate $\partial_{\alpha} \mapsto \partial_{\alpha} - G_{\alpha}^{(1)(2)}\Sigma_{(1)(2)}$ to produce the covariant derivative. If we let $G_{\alpha} = G_{\alpha}^{(1)(2)}\Sigma_{(1)(2)}$, then the gauge field should transform as $G_{\alpha} \mapsto G'_{\alpha} = S_{\mathrm{L}}(\theta)\,G_{\alpha}\,S_{\mathrm{L}}^{-1}(\theta) + S_{\mathrm{L}}(\theta)_{,\alpha}\;S_{\mathrm{L}}^{-1}(\theta)$ under the local gauge transformation $\phi \mapsto \phi' = S_{\mathrm{L}}(\theta)\;\phi$.

Since, in general, $[\Sigma_{(1)(2)\,,\alpha}\,, \Sigma_{(1)(2)}] \neq 0$, the naive procedure does not work and we must introduce additional generators (and gauge fields) so that $\Sigma_{(1)(2)\,,\alpha}$ may be expressed as a linear combination of these generators. This leads to the interpretation of the group of "local little symmetry" transformations as a subset of local $\overline{SO(3,1)}$ gauge transformations (with associated gauge felds $\frac{1}{2}G_{\alpha}^{(\mu)(\nu)}\;E^{\beta}_{(\mu)}\;E^{\rho}_{(\nu)}\;S_{\beta\rho}$ that represent the Levi-Civita spin connection ω_{α}), whose local parameters are $\theta\;\Sigma_{\alpha\beta}$.

Perhaps the simplest approach here is to regard ϕ as not being associated with an electron at all, but as defining an orthogonal frame to which we refer the calculation of another spinor field ϕ' that does correspond to an electron by solving the Dirac equation for ϕ' in this frame. We define a trivial (zero curvature) $\overline{SO(3,1)}$ connection ω_{α} in terms of the spinor $\phi = \lambda - i\gamma^5\epsilon^{-1}\tilde{\xi}$ by

$$\omega_{\alpha} = N\left(\lambda_{,\alpha}\,\xi - \gamma^5\lambda_{,\alpha}\,\xi\gamma^5 + \epsilon^{-1}\tilde{\xi}_{,\alpha}\,\tilde{\lambda}\epsilon - \gamma^5\epsilon^{-1}\tilde{\xi}_{,\alpha}\,\tilde{\lambda}\epsilon\gamma^5\right), \tag{17}$$

where $N = \frac{1}{2}(N_0\gamma_0 - N_5\gamma^5)/(N_0^2 + N_5^2)$. (One can show, after some algebra, that ω does indeed transform as a $\overline{SO(3,1)}$ connection and has zero curvature, but we shall not take the space to prove it here.) Denoting the covariant derivative based on this connection by $D_{\alpha} = \partial_{\alpha} - \omega_{\alpha}$, we find that $D_{\alpha}\phi \equiv 0$, which is analogous to Ricci's lemma in Riemannian geometry. Although the curvature of this connection is zero, this spin connection contains non-trivial information concerning centrifugal and Coriolis accelerations of the frame. (Can one incorporate some form of Mach's Principle into the dynamical formalism using ϕ?) On the other hand, since both the covariant derivative of ϕ and curvature of ω_α are zero, it is not clear how to write down field equations for ϕ.

6. – Conclusion

Classical models of a massive electron with spin that employ spinors to carry the spin degrees of freedom and which respect the Frenkel condition $\Sigma_{\alpha\beta}\dot{x}^\beta = 0$ must possess a doubling of spinor states. The physical basis of this degeneracy has been explained in terms of local little symmetry.

The free-field Dirac spinor wave function has observables that have been shown to be invariant under local little symmetry transformations. The problem of constructing a gauge-invariant Dirac Lagrangian from a gauge covariant derivative has not been completely solved, although one possible solution has been suggested. The Levi-Civita spin connection in this case is constructed from a complex Dirac spinor and must contain inertial information. This raises the question of whether this construction embodies Mach's Principle, which would demand that ϕ somehow describe the rest of the matter in the Universe. However, it is more likely that ϕ and its associated tetrad will turn out to be no more observable (measurable) than are space-time coordinates, since a governing field equation for ϕ seems not to exist.

REFERENCES

[1] DIRAC P. A. M., *J. Math. Phys.*, **4** (1963) 901.
[2] NASH P. L., *J. Math. Phys.*, **25** (1984) 204.
[3] NASH P. L., *J. Math. Phys.*, **25** (1984) 2104.
[4] FRENKEL W. J., *Z. Phys.*, **37** (1926) 243.
[5] THOMAS L. H., *Philos. Mag.*, **3** (1927) 1.
[6] NASH P. L., *Nuovo Cimento B*, **105** (1990) 31.
[7] NASH P. L., *Nuovo Cimento B*, **107** (1992) 1291.