The Feynman path integral quantization of constrained systems

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Summary. — The Feynman path integral for constrained systems is constructed using the canonical formalism introduced by Güler. This approach is applied to a free relativistic particle and Christ-Lee model.

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1. – Introduction

The quantization of a classical system can be achieved by the canonical quantization method [1]. If we ignore the ordering problems, it consists in replacing the classical Poisson bracket, by quantum commutators when classically all the states on the phase space are accessible. This is no longer correct in the presence of constraints. An approach due to Dirac [2] is widely used for quantizing the constrained Hamiltonian systems [3-5].

The path integral is another approach used for the quantization of constrained systems. This approach was formulated by Faddeev [6]. Faddeev and Popov [7] handle constraints in the path integral formalism by quantizing singular theories with first-class constraints in the canonical gauge. The generalization of the method to theories with second-class constraints is given by Senjanovic [8]. Fradkin and Vilkovisky [9, 10] rederived both results in a broader context, where they improved Faddeev's procedure mainly to include covariant constraints; also they extended this procedure to the Grassman variables.

When the dynamical system possesses some second-class constraints there exists another method given by Batalin and Fradkin [11]: the BFV-BRST operator quantization method. One enlarges the phase space in such a way that the original second-class constraints become converted into the first-class ones, so that the number of physical degrees of freedom remains unaltered.

These quantization schemes have the properties that by using them one can easily control important properties of quantum theory such as unitarity and positivedefiniteness of the metric. Besides, relativistically covariant formulation of quantum theory is obtained by the quantization schemes.

Now we would like to make a brief review of the path integral formulation.

2. – The Feynman path integral formulation

The path integral quantization is defined by the Feynman kernel [12, 13].

In the operator version of canonical quantization one turns the functions q^i, p_i into operators $\hat{q}^i$ and $\hat{p}_i$, which satisfy the commutator relations

$$[\hat{q}^k, \hat{p}_\varrho] = i\delta^k_l \,. \tag{1}$$

Eigenstates defined by the eigenvalue equations

$$\hat{q}^i\,|q\rangle = q^i\,|q\rangle, \qquad \hat{p}^i\,|p\rangle = p^i\,|p\rangle, \tag{2}$$

form an orthonormal system

$$\begin{cases} \langle q'\,|q\rangle = \delta(q'-q), & \langle p'\,|p\rangle = \delta(p'-p), \\ \int \mathrm{d}q\,|q\rangle\langle q| = 1\,, & \int \mathrm{d}p\,|p\rangle\langle p| = 1\,. \end{cases} \tag{3}$$

This transition may be performed at any time. States at arbitrary times are obtained by means of unitary transformation generated by the Hamiltonian

$$|q, t\rangle = \exp[it\hat{H}]\,|q\rangle, \tag{4}$$

where $|q\rangle$ is assumed to be an eigenstate at $t=0$. The propagator (Feynman kernel) for the wave function $\psi(q, t) = \langle a, t|\psi\rangle$ is thus

$$\begin{cases} D(q', t', q, t) = \langle q', t'\,|q, t\rangle, \\ D(q', t', q, t) = \langle q'\,|\exp[-i(t'-t)\,\hat{H}]\,|q\rangle. \end{cases} \tag{5}$$

In this case the Feynman kernel connects the Schrödinger wave function in two different times as

$$\psi(q', t') = \int \mathrm{d}q\, D(q', t', q, t)\,\psi(q, t). \tag{6}$$

There are many prescriptions to define the Feynman path integral. This freedom reflects the fact that a classical Hamiltonian does not uniquely determine a quantum Hamiltonian—there is an operator ambiguity. Different path integral definitions correspond to different quantum operator orderings. In our calculations we will use a specific one, the Weyl ordering [14-18], which will be discussed very briefly.

Let us define the momentum and the position operators as

$$\langle q'\,|p|q\rangle = \int \frac{\mathrm{d}p}{2\pi}\,p\exp[ip(q'-q)], \qquad \langle q'\,|q|q\rangle = q\delta(q'-q). \tag{7}$$

The Weyl ordering is defined in the following way:

$$(\widehat{p}\widehat{q})_{\mathrm{w}} = \frac{1}{2}(\widehat{p}\widehat{q} + \widehat{q}\widehat{p}), \tag{8}$$

$$(\widehat{p}\widehat{q}^{3})_{\mathrm{w}} = \frac{1}{4}(\widehat{p}\widehat{q}^{3} + \widehat{q}\widehat{p}\widehat{q}^{2} + \widehat{q}^{2}\widehat{p}\widehat{q} + \widehat{q}^{3}\widehat{p}), \text{ etc.}, \tag{9}$$

$$\text{General expression} = \frac{\sum \text{all possible orders}}{\text{total number of possible orders}}. \tag{10}$$

The above treatment leads us to obtain

$$\langle q'|\widehat{H}_{\mathrm{w}}|q\rangle = \int \frac{\mathrm{d}p}{2\pi} \exp[ip(q'-q)]\, H\left(p, \frac{q'+q}{2}\right), \tag{11}$$

where H_{w} is the Weyl transform of the Hamiltonian operator $\widehat{H}(\widehat{q}, \widehat{p})$. Thus, the Weyl order is specified to be the mid-point prescription.

To clarify the situation we consider the path integral in curvilinear coordinates. Consider the following point canonical transformation:

$$x^{a} \rightarrow q^{a} = f^{a}(x), \qquad a = 1, \ldots, D, \tag{12}$$

$$(\mathrm{d}s)^2 = \sum_a (\mathrm{d}x^a)^2 = \sum_{ab} \mathrm{d}q^a\, \mathrm{d}q^b\, M_{ab}, \tag{13}$$

where the matrix M_{ab} is given by

$$M_{ab} = \frac{\partial x_c}{\partial q_a} \frac{\partial x_c}{\partial q_b}. \tag{14}$$

The volume element in the two representations is given by

$$\mathrm{d}^a x = g\, \mathrm{d}^a q, \tag{15}$$

where

$$g = \det(M_{ab})^{1/2}, \qquad \mathrm{d}^a x = \mathrm{d}x_1 \ldots \mathrm{d}x_D, \qquad \mathrm{d}^a q = \mathrm{d}q_1 \ldots \mathrm{d}q_D. \tag{16}$$

In the q system, since g is evaluated at the mid-point, it cannot be used to make the volume element $\mathrm{d}^a q = \mathrm{d}q_1 \ldots \mathrm{d}q_D$ an invariant, so it is convenient to eliminate the Jacobian factor in the volume element [14-18]. Thus we introduce

$$\langle x|t\rangle = \frac{1}{\sqrt{g}}\langle q|t\rangle \quad \text{and} \quad \langle x|a\rangle = \frac{1}{\sqrt{g}}\langle q|a\rangle. \tag{17}$$

Hence

$$\phi(q, t) = \langle q|t\rangle = \sqrt{g}\,\psi(x, t), \tag{18}$$

$$\phi_a(q) = \langle q|a\rangle = \sqrt{g}\,\psi_a(x). \tag{19}$$

Thus the Weyl transform of the Hamiltonian H is defined as

$$\overline{H} = \sqrt{g}\,(\hat{q})\,\widehat{H}(\hat{q},\,\hat{p})\,\frac{1}{\sqrt{g(\hat{q})}}\,, \tag{20}$$

$$\overline{H} = \widehat{H}(\hat{q},\,\hat{p})_{\mathrm{w}} + \Delta V_{\mathrm{w}}\,(\hat{q})\,. \tag{21}$$

In theories with the Lagrangian given in the form

$$L = \frac{1}{2}\,M_{ab}\dot{q}^a\dot{q}^b - V(q)\,, \tag{22}$$

the Hamiltonian $\widehat{H}(\hat{q},\,\hat{p})_{\mathrm{w}}$ and $\Delta V_{\mathrm{w}}(\hat{q})$ are defined as

$$\widehat{H}(\hat{q},\,\hat{p})_{\mathrm{w}} = \frac{1}{8}\,(\hat{p}_a\hat{p}_b M_{ab}^{-1} + 2\hat{p}_a\hat{p}_b M_{ab}^{-1}\hat{p}_b + \hat{p}_b M_{ab}^{-1}\hat{p}_a\hat{p}_b) + V(\hat{q})\,, \tag{23}$$

$$\Delta V_{\mathrm{w}}(q) = \frac{1}{8}\left[\frac{\partial}{\partial q_a}\left(\frac{\partial q_b}{\partial x_c}\right)\right]\left[\frac{\partial}{\partial q_b}\left(\frac{\partial q_a}{\partial x_c}\right)\right]. \tag{24}$$

Now the path integral representation of the propagator in the phase space is defined as

$$\langle q'\,|\exp[-i(t'-t)\,\widehat{H}]\,|q\rangle = \int(\mathrm{D}q)(\mathrm{D}p)\,\exp\left[i\left[\int(p\dot{q}-\overline{H})\right]\mathrm{d}t\right], \tag{25}$$

where $\overline{H}$ is the Weyl transform of $\widehat{H}(\hat{q},\,\hat{p})$.

In order to obtain the path integral expression in configuration space, we perform p integration in (25)

$$\langle q'\,|\exp[-i(t'-t)\,\widehat{H}]\,|q\rangle = \int(\mathrm{D}q)(g)\,\exp\left[i\left[\int(L(q_i,\,\dot{q}_i,\,t) - \Delta V_{\mathrm{w}}(q))\right]\mathrm{d}t\right]. \tag{26}$$

For the quantization of singular systems, Faddeev [18] incorporated the Dirac formalism into the Hamiltonian form of the Feynman integral. Now we would like to discuss his formulation briefly.

Consider a system with n degrees of freedom. It may have r first-class constraints ϕ^α, but no second-class constraints. Let us choose r gauge constraints χ^α, then $2r$ constraints fulfil

$$\{\phi^\alpha,\,\phi^\beta\} = 0\,, \qquad \alpha,\,\beta = 1,\,2,\,\ldots,\,r\,, \tag{27}$$

$$\det|\{\phi^\alpha,\,\chi^\beta\}| \neq 0 \tag{28}$$

on the hypersurface defined by $\phi^\alpha = 0$, $\chi^\alpha = 0$, where $\{\,,\}$ denotes the Poisson bracket.

The path integral representation is given as

$$\langle q'\,|\exp[-i(t'-t)\,\widehat{H}_0]\,|q\rangle = \int\prod_t \mathrm{d}\mu\,(q_j,\,p_j)\,\exp\left[i\left\{\int_{-\infty}^{+\infty}\mathrm{d}t\,(p_j\dot{p}_j - H_0)\right\}\right], \qquad j = 1,\,\ldots,\,n \tag{29}$$

where the measure of integration is given as

$$\mathrm{d}\mu(q, p) = \det|\{\phi^\alpha, \phi^\beta\}| \prod_{\alpha=1}^{r} \delta(\chi^\alpha)\, \delta(\phi^\alpha) \prod_{j=1}^{n} \mathrm{d}q^j\, \mathrm{d}p_j\,, \tag{30}$$

and the trajectories $q(t)$ coincide at $t \to \pm\infty$ with the solutions $q_{\mathrm{in}}(t)$ and $q_{\mathrm{out}}(t)$ of the equations describing the asymptotic motion.

The expression (29) can be written in an equivalent form,

$$\langle q' | \exp[-i(t'-t)\widehat{H}_0] | q \rangle = \int \prod \mathrm{d}q^*\, \mathrm{d}p^* \exp\left[i \left\{ \int_{-\infty}^{+\infty} \mathrm{d}t\, (p^* \dot{q}^* - H^*) \right\} \right]. \tag{31}$$

In (29) (p_j, q_j) are any set of coordinates while in (31) (p^*, q^*) are canonical coordinates and H^* denotes a Hamiltonian written in terms of q^*'s and p^*'s. In order to prove (31), one changes, coordinates from the set (p, q) to $(p^*, p_\alpha, Q^*, q_\alpha)$ with (p_α, q_α) "redundant variables", such that $p_\alpha = \chi_\alpha$. One gets rid of the redundant variables with delta-functions, and the residue is $\det|\{\phi^\alpha, \phi^\beta\}|$. In fact the functional integral representation (31) is an integration over the independent variables q^*, p^*.

If there exist additional $2r'$ second-class constraints θ_μ, the path integral representation is given by Senjanovic [8] as

$$\langle q' | \exp[-i(t'-t)\widehat{H}_0] | q \rangle = \int \prod_{\alpha}^{r} \det|\phi^\alpha, \phi^\beta\}|\, \delta(\chi^\alpha)\, \delta(\phi^\alpha) \cdot \tag{32}$$

$$\cdot \prod_{\mu}^{2r'} \delta(\theta_\mu) \det\{\theta_\alpha, \theta_\beta\}|^{1/2} \prod_{j} \mathrm{d}q^j\, \mathrm{d}p_j \exp\left[i \left\{ \int_{-\infty}^{+\infty} \mathrm{d}t\, (p_j \dot{q}_j - H_0) \right\} \right].$$

Another approach on the Feynman path integral quantization of constrained systems is discussed by Blau [18]. In this approach, Blau writes down the Feynman path integrals as follows: given a classical Hamiltonian, one constructs a quantum Hamiltonian by the usual procedure of promoting the position and the momentum functions to quantum operators. In the presence of constraints, the quantum Hamiltonian acts on a restricted Hilbert space. In this case one can reexpress the Hamiltonian in terms of canonical position and momentum operators in the restricted Hilbert space. Then one can make a correspondence between these canonical operators and the classical functions which appear in the path integral. In this case the path integral representation is given as

$$\langle q' | \exp[-i(t'-t)\widehat{H}] | \overline{q} \rangle = \int_{q}^{q'} (\mathrm{D}q^*)(\mathrm{D}p^*) \exp\left[\left\{ i \int_{t}^{t'} (p^* \dot{q}^* - H')\, \mathrm{d}t \right\} \right], \tag{33}$$

where q^*, p^* are the canonical phase space and H' denotes a Hamiltonian written by q^*'s and p^*'s.

3. – The canonical formulation

The canonical formulation [19-21] gives the set of Hamilton-Jacobi partial-differential equation (HJPDE) as

$$H'_\alpha\left(t_\beta,\, q_a,\, \frac{\partial S}{\partial q_a},\, \frac{\partial S}{\partial t_\alpha}\right) = 0\,, \qquad \alpha,\,\beta = 0,\, n-r+1,\, \dots,\, n\,, \quad a = 1,\, \dots,\, n-r\,, \tag{34}$$

where

$$H'_\alpha = H_\alpha(t_\beta,\, q_a,\, p_a) + p_\alpha\,, \tag{35}$$

and H_0 is defined as

$$H_0 = -L(t,\, q_i,\, \dot{q}_\nu,\, \dot{q}_a = w_a) + p_a w_a + \dot{q}_\mu p_\mu \big|_{p_\nu = -H_\nu}\,, \qquad \nu = 0,\; n-r+1,\, \dots,\, n\,. \tag{36}$$

The equations of motion are obtained as total differential equations in many variables as follows:

$$\mathrm{d}q_a = \frac{\partial H'_\alpha}{\partial p_a}\,\mathrm{d}t_\alpha\,, \quad \mathrm{d}p_a = \frac{\partial H'_\alpha}{\partial p_a}\,\mathrm{d}t_\alpha\,, \quad \mathrm{d}p_\mu = -\frac{\partial H'_\alpha}{\partial t_\mu}\,\mathrm{d}t_\alpha\,, \qquad \mu = 1,\, \dots,\, r\,, \tag{37}$$

$$\mathrm{d}z = \left(-H_\alpha + p_a\,\frac{\partial H'_\alpha}{\partial p_a}\right)\mathrm{d}t_\alpha\,, \tag{38}$$

where $z = S(t_\alpha,\, q_a)$. The set of eqs. (37), (38) is integrable if

$$\mathrm{d}H'_0 = 0\,, \tag{39}$$

$$\mathrm{d}H'_\mu = 0\,, \qquad \mu = 1,\, \dots,\, r\,. \tag{40}$$

If conditions (39) and (40) are not satisfied identically, one considers them as new constraints and again tests the consistency conditions. Thus, repeating this procedure one may obtain a set of conditions.

Now we would like to give the Feynman path integral formulation in the canonical method. The canonical formalism leads us to obtain the set of canonical phase space coordinates q_a and p_a as functions of t_α, besides the canonical action integral is obtained in terms of the canonical coordinates. H'_α can be interpreted as the infinitesimal generators of the canonical transformation given by parameters t_α, respectively. In this case, the propagator for the constrained system is given as

$$D(q'_a,\, t'_\alpha;\, q_a,\, t_\alpha) = \int_{q_a}^{q'_a} (\mathrm{D}q^a)(\mathrm{D}p^a)\,\exp\left[i\left\{\int_{t_\alpha}^{t'_\alpha}\left(-\overline{H}_\alpha + p_a\,\frac{\partial \overline{H}_\alpha}{\partial p_a}\right)\mathrm{d}t_\alpha\right\}\right], \tag{41}$$

or, in equivalent form,

$$D(q'_a,\, t'_\alpha;\, q_a,\, t_\alpha) = \int_{q_a}^{q'_a} (\mathrm{D}q^a)(\mathrm{D}p^a)\,\exp\left[i\left\{\int_{t_\alpha}^{t'_\alpha}(-\overline{H}_\alpha\,\mathrm{d}t_\alpha + p_a\,\mathrm{d}q_a)\right\}\right], \tag{42}$$

$$a = 1,\, \dots,\, n-r,\quad \alpha = 0,\, n-r+1,\, \dots,\, n\,,$$

where $\overline{H}_\alpha$ are Weyl-ordered transform of Hamiltonian operators $\widehat{H}_\alpha(t_\beta,\, \widehat{q}_a,\, \widehat{p}_a)$.

In the following two sections we will work out the Feynman path integral for two singular systems: the free relativistic particle and the Christ-Lee model.

3·1. *The Feynman path integral for a free relativistic particle.* – As a first example let us consider a free relativistic particle of non-zero mass, moving in D-dimensional Minkowski space described by the usual parametrization-invariant action [3]. The action is given as

$$S = -m\int(-\dot{x}_\mu \dot{x}^\mu)^{1/2}\,\mathrm{d}\tau\,, \qquad \mu = 0, 1, \dots, , D-1\,. \tag{43}$$

Here, x^μ are functions of arbitrary parameter τ describing the displacement of the particle along its world line and the Lagrangian is

$$L = -m(-\dot{x}_\mu \dot{x}^\mu)^{1/2}\,. \tag{44}$$

The Lagrangian L is singular since its Hessian

$$A_{\mu\nu} = \frac{\partial^2 L}{\partial \dot{x}^\mu\,\partial \dot{x}^\nu} = -\frac{m}{(-\dot{x}^2)^{3/2}}(g_{\mu\nu}\dot{x}^2 - \dot{x}_\mu \dot{x}_\nu)\,, \tag{45}$$

has rank $D-1$. $\mu, \nu = 1, 2, \dots, D-1$ and the metric convention is “mostly plus”.

The generalized momenta p_μ conjugate to the coordinate x^μ are defined as

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{(-\dot{x}^2)^{1/2}}\,. \tag{46}$$

Therefore the zeroth component is

$$p_0 = \frac{m\dot{x}_0}{(-\dot{x}^2)^{1/2}}\,, \tag{47}$$

and the i-th component is

$$p_i = \frac{m\dot{x}_i}{(-\dot{x}^2)^{1/2}}\,, \qquad i = 1, \dots, D-1\,. \tag{48}$$

Since the rank of the Hessian matrix is $D-1$, one can solve (48) for $\dot{x}^i$ in terms of p^i and $\dot{x}^0$. In fact

$$\dot{x}^i = \frac{p^i \dot{x}^0}{(m^2 + p_i^2)^{1/2}} = \omega^i\,. \tag{49}$$

Substituting (49) in (47), one has

$$p^0 = \left.\frac{m\dot{x}^0}{(-\dot{x}^2)^{1/2}}\right|_{\dot{x}^i = \omega^i}\,. \tag{50}$$

Thus, we obtain

$$p^0 = (m^2 + p_i^2)^{1/2}\,. \tag{51}$$

Hence, the primary constraint is

(52) $$H_0' = p_0 + H_0 = 0\,,$$

where H_0 is defined as

(53) $$H_0 = (m^2 + p_i^2)^{1/2}\,.$$

Besides, the canonical Hamiltonian is defined as

(54) $$H = -L(x^0, x^i, \dot{x}^0, \omega^i) - (m^2 + p_i^2)^{1/2}\dot{x}^0 + p_i\omega^i\,.$$

Calculations show that H vanishes identically.

The canonical method [19-21] leads us to obtain the set of Hamilton Jacobi partial differential equations as [22]

(55) $$H' = 0\,,$$

(56) $$H_0' = p_0 + H_0\,,$$

where H_0 is defined as

(57) $$H_0 = (m^2 + p_i^2)^{1/2}\,, \qquad i = 1, \ldots, D-1\,.$$

Making use of (35), (37) and (55), (56), the phase space coordinates x^i and p^i are obtained in terms of x^0. Besides, the canonical action is calculated as

(58) $$z = \int_{x^{0\prime}}^{x^{0\prime\prime}} \left(-H_0 + p_i\,\frac{\partial H_0'}{\partial p_i}\right) \mathrm{d}x^0\,,$$

or

(59) $$z = \int_{x^{0\prime}}^{x^{0\prime\prime}} (-H_0 + p_i\dot{x}_i)\,\mathrm{d}x^0\,.$$

Making use of (42) and (59), the path integral for a single relativistic particle is expressed as

(60) $$\langle x'', x^{0\prime\prime} | x', x^{0\prime}\rangle = \int \mathrm{d}^{D-1}x\,\mathrm{d}^{D-1}p\,\exp\left[i\left\{\int_{x^{0\prime}}^{x^{0\prime\prime}} (-\overline{H}_0 + p_i\dot{x}_i)\,\mathrm{d}x^0\right\}\right],$$

where $\overline{H}_0$ is the Weyl transform of the Hamiltonian $\widehat{H}_0$ and it is given as

(61) $$\overline{H}_0 = (m^2 + p_i^2)^{1/2}\,.$$

Note that, in four-dimensional Minkowski space, one has

(62) $$\langle x'', t'' | x', t'\rangle = \int \mathrm{d}^3x\,\mathrm{d}^3p\,\exp\left[i\left\{\int_{t'}^{t''} (p\dot{x} - (m^2 + p^2)^{1/2})\,\mathrm{d}t\right\}\right],$$

in agreement with [23].

3·2. *The Feynman path integral for the Christ-Lee model.* – As a second example we consider the Christ-Lee problem [14, 15, 24], which is described by the singular Lagrangian

$$L = \frac{1}{2}\,(\dot{r}^2 + r^2(\dot{\theta} - \lambda)^2) - V(r)\,. \tag{63}$$

Here, r and θ are polar coordinates, λ is another generalized coordinate. $V(r)$ is the central potential of the system. The generalized momenta read as

$$p_r = \dot{r}\,, \tag{64}$$

$$p_\theta = r^2(\dot{\theta} - \lambda)^2\,, \tag{65}$$

$$p_\lambda = 0\,. \tag{66}$$

Since the rank of the Hessian matrix is two, we have only one primary constraint as

$$H'_\lambda = 0\,. \tag{67}$$

The canonical Hamiltonian H_0 reads as

$$H_0 = \frac{p_{r^2}}{2} + \frac{p_{\theta^2}}{2r^2} + \lambda p_\theta + V(r)\,. \tag{68}$$

Equations (67) and (68) lead to the set of Hamilton-Jacobi partial-differential equations

$$H'_0 = p_0 + H_0 = 0\,, \tag{69}$$

$$H'_\lambda = p_\lambda = 0\,. \tag{70}$$

Making use of (42) and (69), (70), the path integral for this system is obtained as

$$\langle r',\, \theta',\, \lambda',\, t';\, r,\, \theta,\, \lambda,\, t\rangle = \int \mathrm{D}r\,\mathrm{D}\theta\,\mathrm{D}p_r\,\mathrm{D}p_\theta\,\exp\left[i\left\{\int_t^{t'}(-\overline{H}_0 + p_r\dot{r} + p_\theta\dot{\theta})\,\mathrm{d}t\right\}\right], \tag{71}$$

where $\overline{H}_0$ is the Weyl-ordered transform of the Hamiltonian H_0 which can be obtained as

$$\overline{H}_0 = \sqrt{r}\,\widehat{H}_0(\widehat{p}_r,\, \widehat{p}_\theta,\, \widehat{r},\, \widehat{\theta},\, \lambda)\,\frac{1}{\sqrt{r}}\,, \tag{72}$$

$$\overline{H}_0 = \frac{p_{r^2}}{2} + \frac{p_{\theta^2}}{2r^2} + \lambda p_\theta + V(r) - \frac{1}{8r^2}\,. \tag{73}$$

An important point to be specified here is that from the set of (HJPDE) and the equations of motion, the Hamiltonians H'_0 and H'_λ are interpreted as infinitesimal generators of canonical transformations for two parameters t, λ, respectively. Although λ is introduced as a coordinate in the Lagrangian, the integrability conditions $\mathrm{d}H'_0 = 0$ and $\mathrm{d}H'_\lambda = 0$ force us to treat it as a parameter like t.

Equation (71) may be expressed as

$$(74)\quad \langle r', \theta', \lambda', t'; r, \theta, \lambda, t\rangle = \lim_{\varepsilon\to 0}\int_q^{q'}\prod_{n=1}^{N}\frac{\mathrm{d}r(n)\,\mathrm{d}\theta(n)\,\mathrm{d}p_r(n)\,\mathrm{d}p_\theta(n)}{(2\pi)^2}\cdot$$

$$\cdot\exp\left[i\varepsilon\left\{p_{r_n}\dot{r}_n - \frac{p_{r_n}^2}{2} - V(\bar{r}_n) + \frac{1}{8\bar{r}_n^2} + p_{\theta_n}(\dot{\theta}-\lambda) - \frac{p_{\theta_n^2}}{2\bar{r}_n^2}\right\}\right],$$

where

$$(75)\quad \begin{cases} q_n = \{r_n, \theta_n\}, \quad \dot{q}_n = \dfrac{q_{n+1}-q_n}{\varepsilon}, \quad \bar{q}_n = \dfrac{q_{n+1}+q_n}{2}, \\ q_{N+1} = q', \quad N\varepsilon = t'-t. \end{cases}$$

Integrating over p_θ we obtain

$$(76)\quad \langle r', \theta', \lambda', t'; r, \theta, \lambda, t\rangle = \lim_{\varepsilon\to 0}\int_q^{q'}\prod_{n=1}^{N}\frac{\bar{r}\,\mathrm{d}r(n)\,\mathrm{d}\theta(n)\,\mathrm{d}p_r(n)\,\mathrm{d}p_\theta(n)}{(2\pi)}\cdot$$

$$\cdot\left(\frac{1}{2i\pi\varepsilon}\right)^{1/2}\exp\left[\frac{i\varepsilon}{2}[\bar{r}(\dot{\theta}-\lambda)]^2\right]\exp\left[i\varepsilon\left\{p_r\dot{r} - \frac{p_r^2}{2} - V(\bar{r}_n) + \frac{1}{8\bar{r}^2}\right\}\right].$$

As was specified previously that t, λ are two independent parameters, we will evaluate the path integral for a given value of λ. For simplicity let us take $\lambda = 0$. In this case (76), after integration over θ, will give

$$(77)\quad \langle r', t'; r, t\rangle = \lim_{\varepsilon\to 0}\int_q^{q'}\prod_{n=1}^{N}\frac{\mathrm{d}r(n)\,\mathrm{d}p_r(n)}{(2\pi)}\exp\left[i\varepsilon\left\{p_r\dot{r} - \frac{p_{r^2}}{2} - V(\bar{r}) + \frac{1}{8\bar{r}^2}\right\}\right].$$

Now integrating over p_r we obtain

$$(78)\quad \langle r', t'; r, t\rangle = \int \mathrm{D}r\exp\left[i\int\left\{\frac{\dot{r}^2}{2} - V(\bar{r}) + \frac{1}{8r^2}\right\}\mathrm{d}t\right],$$

This result is in complete agrement with the results given in ref. [15, 18, 23].

4. – Conclusion

In this work we have followed the canonical method to construct the Feynman path integral for constrained systems. This treatment leads us to the equations of motion as total differential equations in many variables. If the system is integrable, then one can construct the canonical variables q_a and p_a in terms of t_α. Besides the action integral is obtained in terms of the canonical variables with the Weyl transform of the Hamiltonian operators $\widehat{H}_\alpha$.

It is obvious from eq. (42) that our approach has two advantages: first, we avoid to solve explicitly the higher-order generation constraints. Second, in the case of relativistic theory, the exponent in the path integral is manifestly invariant, while secondary- or higher-generation constraints usually spoil manifestly Lorentz invariance.

In the relativistic-particle example, since the integrability conditions $\mathrm{d}H' = 0$, $\mathrm{d}H'_0 = 0$ are satisfied identically, the system is integrable. Hence, the canonical phase space coordinates x^i and p_i are obtained in terms of the parameter (x^0). The path integral is then followed directly as given in (60). In usual formulation [23], one has to fix a gauge and to integrate over the extended phase space and after integration over the redundant variables, one can arrive at the result (62).

The Christ-Lee system is integrable. Hence, H'_0 and H'_λ can be interpreted as infinitesimal generators of canonical transformations given by the parameters t and λ. Although λ is introduced as a coordinate in the Lagrangian, the presence of constraints and the integrability conditions force us to treat it as a parameter like t. In this case the path integral is obtained as an integration over the canonical phase space coordinates r, θ, p_r, p_θ. Other treatments [15, 23] need a gauge-fixing condition to obtain the path integral over the canonical variables.

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