Hawking quantum thermal effect of non-static spherically symmetric space-time and non-static plane-symmetric space-time (*)

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Summary. — In this article, we prove that at time coordinate v_*, through the generalized tortoise-type coordinate transformation, the Klein-Gordon equation in non-static plane-symmetric space-time and non-static spherically symmetric space-time will be changed into the standard motion equation which is near the horizon. Then the equation deciding the location of the event horizon is obtained. Further, through studying the wave function, the Hawking radiation temperature is obtained. This shows that the Hawking radiation temperature is a compensate effect under the time scale transformation.

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The common way of studying the Hawking effect of a non-static black hole is in terms of quantum field theory of curved space-time [1-4] (the back-reaction of thermal radiation is considered). This way is complicated, and can only show the approximate value of the temperature and the location of the event horizon in the circumstances of weak radiation. In this article, the difficulty of calculating energy-momentum tensors is avoided; we proceed directly from discussing the Klein-Gordon equation which is near the horizon, through the generalized tortoise-type coordinates and the time scale transformation, thus obtaining the equations deciding the event horizon location. Further, through discussing the wave function, the Hawking radiation spectrum and Hawking radiation temperature are obtained. By analysing the temperature of two different time coordinates, we think the Hawking radiation temperature is a compensate effect under the time scale transformation.

We use the system of units $C = \hbar = G = K_B = 1$.

The line element of a spherically symmetric or plane-symmetric non-static space-time [5] is

$$ds^2 = g_{00}\, dv^2 + 2g_{01}\, dv\, dx + g_{22}\, dy^2 + g_{33}\, dz^2 , \tag{1}$$

this metric determinant is $g = -g_{01}^2 g_{22} g_{33}$.

(*) The authors of this paper have agreed to not receive the proofs for correction.

The non-zero contravariant metric tensor is

$$g^{01} = 1/g_{01}\,, \qquad g^{11} = -g_{00}/g_{01}^2\,, \qquad g^{22} = 1/g_{22}\,, \qquad g^{33} = 1/g_{33}\,. \tag{2}$$

If this space-time has event horizon F, it satisfies the null-surface equation [6]

$$g^{\mu\nu} \frac{\partial F}{\partial x^\mu} \frac{\partial F}{\partial x^2} = 0\,. \tag{3}$$

Taking into account its spherically symmetric and plane-symmetric character, eq. (3) can be reduced to

$$2g^{01} \frac{\partial F}{\partial v} \frac{\partial F}{\partial x} + g^{11} \left(\frac{\partial F}{\partial x} \right)^2 = 0\,, \tag{4}$$

that is

$$g^{11} - 2g^{01}\dot{x} = 0 \tag{5}$$

or

$$g_{00} + 2g_{01}\dot{x} = 0\,, \tag{6}$$

where

$$\dot{x} = \frac{\partial x}{\partial v}\bigg|_{y,\,z} = -\frac{\partial F}{\partial v} \bigg/ \frac{\partial F}{\partial x}\,.$$

Assuming $x = x_{\mathrm{EH}}$ is a solution of eq. (6), that is the location of the event horizon.

The Klein-Gordon equation of space-time (1) is

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g}\, g^{\mu\nu} \frac{\partial \psi}{\partial x^\nu} \right) - \mu^2 \psi = 0\,. \tag{7}$$

It can be reduced to

$$2\sqrt{-g}\, g^{01} \frac{\partial^2 \psi}{\partial v\, \partial x} + \frac{\partial}{\partial v} (\sqrt{-g}\, g^{01}) \frac{\partial \psi}{\partial x} + \frac{\partial}{\partial x} (\sqrt{-g}\, g^{10}) \frac{\partial \psi}{\partial v} + \tag{8}$$

$$+ \frac{\partial}{\partial x} (\sqrt{-g}\, g^{11}) \frac{\partial \psi}{\partial x} + \sqrt{-g}\, g^{11} \frac{\partial^2 \psi}{\partial x^2} + B = 0\,,$$

where

$$B = \frac{\partial}{\partial y} \left(\sqrt{-g}\, g^{22} \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\sqrt{-g}\, g^{33} \frac{\partial \psi}{\partial z} \right) - \sqrt{-g}\, \mu^2 \psi\,,$$

μ is the KG particle quality.

Given the generalized tortoise-type coordinate transformation

$$x_* = \ln[x - x_{\mathrm{EH}}(v)]\,, \qquad v_* = \int A(x_{\mathrm{EH}}, v)\, \mathrm{d}v\,, \tag{9}$$

where

$$A(x_{\mathrm{EH}}, v) = -\left[\frac{g'_{00} + 2\,\dot{x}_{\mathrm{EH}}\,g'_{01}}{g_{01}}\right]_{x = x_{\mathrm{EH}}}, \qquad g'_{00} = \frac{\partial g_{00}}{\partial x}\,, \quad g'_{01} = \frac{\partial g_{01}}{\partial x}\ .$$

Then

$$\text{(10)} \quad \begin{cases} \dfrac{\partial}{\partial x} = \dfrac{1}{x - x_{\mathrm{EH}}}\,\dfrac{\partial}{\partial x_*}\,, \qquad \dfrac{\partial}{\partial v} = A(x_{\mathrm{EH}}, v)\,\dfrac{\partial}{\partial v_*} - \dfrac{\dot{x}_{\mathrm{EH}}}{x - x_{\mathrm{EH}}}\,\dfrac{\partial}{\partial x_*}\,, \\ \dfrac{\partial^2}{\partial x^2} = \dfrac{1}{(x - x_{\mathrm{EH}})^2}\,\dfrac{\partial^2}{\partial x_*^2} - \dfrac{1}{(x - x_{\mathrm{EH}})^2}\,\dfrac{\partial}{\partial x_*}\,, \\ \dfrac{\partial}{\partial v\,\partial x} = -\dfrac{\dot{x}_{\mathrm{EH}}}{(x - x_{\mathrm{EH}})^2}\,\dfrac{\partial^2}{\partial x_*^2} + \dfrac{A(x_{\mathrm{EH}}, v)}{x - x_{\mathrm{EH}}}\,\dfrac{\partial^2}{\partial v_*\,\partial x_*} + \dfrac{\dot{x}_{\mathrm{EH}}}{(x - x_{\mathrm{EH}})^2}\,\dfrac{\partial}{\partial x_*}\ . \end{cases}$$

Equation (8) can be reduced to

$$\text{(11)} \quad \left[\frac{g^{11}}{g^{01}(x - x_{\mathrm{EH}})} - \frac{2\,\dot{x}_{\mathrm{EH}}}{x - x_{\mathrm{EH}}}\right]\frac{\partial^2\psi}{\partial x_*^2} + $$

$$+ \frac{x - x_{\mathrm{EH}}}{\sqrt{-g}\,g^{01}}\left(B + \frac{\partial}{\partial x}(\sqrt{-g}\,g^{10})\,A(x_{\mathrm{EH}}, v)\,\frac{\partial\psi}{\partial v_*}\right) +$$

$$+ 2A(x_{\mathrm{EH}}, v)\,\frac{\partial^2\psi}{\partial v_*\,\partial x_*} + \left[\frac{2\,\dot{x}_{\mathrm{EH}} - g^{11}g_{01}}{x - x_{\mathrm{EH}}} + \frac{1}{\sqrt{-g}\,g^{01}}\,\frac{\partial}{\partial x}(\sqrt{-g}\,g^{11}) + \right.$$

$$\left. + \frac{1}{\sqrt{-g}\,g^{01}}\left(\frac{\partial}{\partial v}(\sqrt{-g}\,g^{01}) - \dot{x}_{\mathrm{EH}}\,\frac{\partial}{\partial x}(\sqrt{-g}\,g^{01})\right)\right]\frac{\partial j}{\partial x_*} = 0\ .$$

Now we search for the asymptotic expression of eq. (11) at $x \to x_{\mathrm{EH}}$.

Analysing the character of the coefficient of $\partial^2\psi/\partial x_*^2$ when $x \to x_{\mathrm{EH}}$ we have

$$\text{(12)} \qquad \lim_{x \to x_{\mathrm{EH}}} \frac{g^{11} - 2\,\dot{x}_{\mathrm{EH}}\,g^{01}}{g^{01}(x - x_{\mathrm{EH}})}\ .$$

If the limit of (12) exists, then we will have

$$\text{(13)} \qquad \lim_{x \to x_{\mathrm{EH}}} (g^{11} - 2\,\dot{x}_{\mathrm{EH}}\,g^{01}) \to 0\ .$$

Using l'Hospital's law and (2), we have

$$\text{(14)} \qquad \lim_{x \to x_{\mathrm{EH}}} \frac{g^{11} - 2\,\dot{x}_{\mathrm{EH}}\,g^{01}}{g^{01}(x - x_{\mathrm{EH}})} = -\frac{g'_{00} + 2\,\dot{x}_{\mathrm{EH}}\,g'_{01}}{g_{01}}\bigg|_{x = x_{\mathrm{EH}}} = A(x_{\mathrm{EH}}, v)\,,$$

where (13) is an equation deciding the location of the event horizon

$$\text{(15)} \qquad (g^{11} - 2\,\dot{x}_{\mathrm{EH}}\,g^{01})_{x = x_{\mathrm{EH}}} = 0\ ;$$

when $x \to x_{\mathrm{EH}}$, the limit of the coefficient of $\partial\psi/\partial x_*$ is

$$(16)\qquad \Delta = \lim_{x\to x_{\mathrm{EH}}} \left\{ \frac{g_{00} + 2\,\dot{x}_{\mathrm{EH}}\, g_{01}}{g_{01}(x - x_{\mathrm{EH}})} + \frac{1}{\sqrt{-g}\, g^{01}} \cdot \right.$$

$$\left. \cdot \left[\frac{\partial}{\partial v} (\sqrt{-g}\, g^{01}) + \dot{x}_{\mathrm{EH}} \frac{\partial}{\partial x} (\sqrt{-g}\, g^{01}) - 2\,\dot{x}_{\mathrm{EH}} \frac{\partial}{\partial x} (\sqrt{-g}\, g^{01}) + \frac{\partial}{\partial x} (\sqrt{-g}\, g^{11}) \right] \right\} =$$

$$= \frac{g'_{00} + 2\,\dot{x}_{\mathrm{EH}}\, g'_{01}}{g_{01}} \Big|_{x = x_{\mathrm{EH}}} + L + \lim_{x\to x_{\mathrm{EH}}} \frac{g^{11} - 2\,\dot{x}_{\mathrm{EH}}\, g^{01}}{2 g^{01} g} \frac{\partial g}{\partial x} +$$

$$+ \lim_{x\to x_{\mathrm{EH}}} \frac{1}{g^{01}} \left[-(g^{01})^2 \frac{\partial g_{00}}{\partial x} - 2 g_{00}\, g^{01} \frac{\partial g^{01}}{\partial x} - 2\,\dot{x}_{\mathrm{EH}} \frac{\partial g^{01}}{\partial x} \right] =$$

$$= -A(x_{\mathrm{EH}}, v) + L + \lim_{x\to x_{\mathrm{EH}}} \left(\frac{-g'_{00} - 2\,\dot{x}_{\mathrm{EH}}\, g'_{01}}{g_{01}} + \frac{2 g_{00}\, g'_{01}}{(g_{01})^2} + \frac{4\,\dot{x}_{\mathrm{EH}}\, g'_{01}}{g_{01}} \right) =$$

$$= -A(x_{\mathrm{EH}}, v) + L + A(x_{\mathrm{EH}}, v) + \lim_{x\to x_{\mathrm{EH}}} \frac{2}{(g_{01})^2} (g_{00}\, g'_{01} + 2\,\dot{x}_{\mathrm{EH}}\, g_{01}\, g'_{01}) =$$

$$= L + \lim_{x\to x_{\mathrm{EH}}} \frac{2 g'_{01}}{(g_{01})^2} (g_{00} + 2\,\dot{x}_{\mathrm{EH}}\, g_{01}) = L\,,$$

where

$$(17)\qquad L = \lim_{x\to x_{\mathrm{EH}}} \frac{1}{\sqrt{-g}\, g^{01}} \left[\frac{\partial}{\partial v} (\sqrt{-g}\, g^{01}) + \dot{x}_{\mathrm{EH}} \frac{\partial}{\partial x} (\sqrt{-g}\, g^{01}) \right] =$$

$$= \left[\frac{\partial}{\partial v} \ln (\sqrt{-g}\, g^{01}) + \dot{x}_{\mathrm{EH}} \frac{\partial}{\partial x} \ln (\sqrt{-g}\, g^{01}) \right]_{x = x_{\mathrm{EH}}}.$$

So eq. (11) can be reduced to

$$(18)\qquad A(x_{\mathrm{EH}}, v) \frac{\partial^2 \psi}{\partial x_*^2} + 2A(x_{\mathrm{EH}}, v) \frac{\partial^2 \psi}{\partial x_*\, \partial v_*} + L \frac{\partial \psi}{\partial x_*} = 0\,.$$

From eq. (2)

$$(19)\qquad \sqrt{-g}\, g^{01} = \sqrt{g_{22}\, g_{33}}\,.$$

By considering its plane-symmetric character,

$$(20)\qquad g_{22} = g_{33} = 1\,.$$

By substituting (20) for (17) we have $L = 0$.

Now the KG equation can be reduced to

$$(21)\qquad \frac{\partial^2 \psi}{\partial x_*^2} + 2 \frac{\partial^2 \psi}{\partial v_*\, \partial x_*} = 0\,.$$

By considering its spherically symmetric character,

$$x = r\,, \qquad y = \theta\,, \qquad z = \phi\,, \qquad g_{22} = r^2\,, \qquad g_{33} = r^2 \sin^2\theta\,.$$

Then $L = 2\,\dot r_{\rm EH}/r_{\rm EH}$.

Now the KG equation can be reduced to

$$A(r_{\rm EH}, v)\,\frac{\partial^2\psi}{\partial r_*^2} + 2A(r_{\rm EH}, v)\,\frac{\partial^2\psi}{\partial r_*\,\partial v_*} + \frac{2\,\dot r_{\rm EH}}{r_{\rm EH}}\,\frac{\partial\psi}{\partial r_*} = 0\,. \tag{22}$$

By considering its spherically symmetric character, the wave function ψ can be written as

$$\psi = \varrho(v, x, y, z)/x\,; \tag{23}$$

when eq. (23) substitutes for (8), we reduce eq. (22) and notice

$$\frac{\partial\psi}{\partial x} = -\frac{\varrho}{x^2} + \frac{1}{x}\,\frac{\partial\varrho}{\partial x}\,, \qquad \frac{\partial\psi}{\partial v} = \frac{1}{x}\,\frac{\partial\varrho}{\partial v}\,,$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{2\varrho}{x^3} - \frac{2}{x^2}\,\frac{\partial\varrho}{\partial x} + \frac{1}{x}\,\frac{\partial^2\varrho}{\partial x^2}\,, \qquad \frac{\partial^2\psi}{\partial v\,\partial x} = -\frac{1}{x^2}\,\frac{\partial\varrho}{\partial v} + \frac{1}{x}\,\frac{\partial^2\varrho}{\partial v\,\partial x}\,.$$

Then the KG equation can be reduced to

$$\begin{aligned} 2\sqrt{-g}\,g^{01}\,\frac{\partial^2\varrho}{\partial v\,\partial x} &+ \left[\frac{\partial}{\partial v}(\sqrt{-g}\,g^{01}) + \frac{\partial}{\partial x}(\sqrt{-g}\,g^{11}) - \frac{2\sqrt{-g}\,g^{11}}{x}\right]\frac{\partial\varrho}{\partial x} + \\ &+ \left[\frac{\partial}{\partial x}(\sqrt{-g}\,g^{10}) - \frac{2\sqrt{-g}\,g^{01}}{x}\right]\frac{\partial\varrho}{\partial v} + \sqrt{-g}\,g^{11}\,\frac{\partial^2\varrho}{\partial x^2} + D = 0\,, \end{aligned} \tag{24}$$

where

$$D = xB - \left[\frac{\partial}{\partial x}(\sqrt{-g}\,g^{01}) + \frac{\partial}{\partial x}(\sqrt{-g}\,g^{11}) - \frac{1\sqrt{-g}\,g^{11}}{x}\right]\frac{\varrho}{x}\,.$$

Under the generalized tortoise-type coordinates eq. (24) can be reduced to

$$\begin{aligned} &\left[\frac{g^{11} - 2\,\dot x_{\rm EH}\,g^{01}}{g^{01}(x - x_{\rm EH})}\right]\frac{\partial^2\varrho}{\partial x_*^2} + 2A(x_{\rm EH}, v)\,\frac{\partial^2\varrho}{\partial v_*\,\partial x_*}\,\cdot \\ &\quad\cdot\frac{x - x_{\rm EH}}{\sqrt{-g}\,g^{01}}\left[D + A(x_{\rm EH}, v)\left(\frac{\partial}{\partial x}(\sqrt{-g}\,g^{01}) - \frac{2\sqrt{-g}\,g^{01}}{x}\right)\frac{\partial\varrho}{\partial v_*}\right] + \\ &\quad + \left\{\frac{2\,\dot x_{\rm EH} - g^{11}g_{01}}{x - x_{\rm EH}} + \frac{1}{\sqrt{-g}\,g^{01}}\left[\frac{\partial}{\partial v}(\sqrt{-g}\,g^{01}) + \frac{\partial}{\partial x}(\sqrt{-g}\,g^{11}) - \right.\right. \\ &\quad \left.\left. - \frac{2\sqrt{-g}\,g^{11}}{x} - \dot x_{\rm EH}\left(\frac{\partial}{\partial x}(\sqrt{-g}\,g^{01}) - \frac{2\sqrt{-g}\,g^{01}}{x}\right)\right]\right\}\frac{\partial\varrho}{\partial x_*} = 0\,. \end{aligned} \tag{25}$$

Equation (25) is compared with eq. (11), and we obtain that $A(x_{\rm EH}, v)$ is the coefficient of $\partial^2\varrho/\partial x_*^2$ and $\partial^2\psi/\partial x_*^2$.

The coefficient of $\partial\varrho/\partial x_*$ is G larger than the coefficient of $\partial\psi/\partial x_*$, where

$$G = \frac{1}{\sqrt{-g}\, g^{01}}\left(-\frac{2\sqrt{-g}\, g^{11}}{x} + \dot{x}_{\rm EH}\frac{2\sqrt{-g}\, g^{01}}{x}\right).$$

Let us calculate the limit of G:

$$\lim_{x\to x_{\rm EH}}\frac{2}{xg^{01}}(-g^{11}+\dot{x}_{\rm EH}g^{01}) = -\lim_{x\to x_{\rm EH}}\frac{2}{xg^{01}}(g^{11}-2\dot{x}_{\rm EH}g^{01}+\dot{x}_{\rm EH}g^{01}) = -\frac{2\dot{x}_{\rm EH}}{x_{\rm EH}}\,; \tag{26}$$

when $x\to x_{\rm EH}$ eq. (25) can be reduced to

$$\frac{\partial^2\varrho}{\partial x_*^2} + 2\frac{\partial^2\varrho}{\partial v_*\,\partial x_*} = 0\,. \tag{27}$$

The form of eq. (27) is the same as eq. (21).

After separating the variables as

$$\varrho = f(x_*, v_*)\, Y(y, z)\,, \tag{28}$$

the radial-part solution of eq. (27) is the ingoing wave solution

$$f_{\rm in} = e^{-i\omega v_*} \tag{29}$$

and the outgoing wave solution is

$$f_{\rm out} = e^{-i\omega v_*}e^{2i\omega x_*} = e^{-i\omega v_*}(x-x_{\rm EH})^{2i\omega}\,. \tag{30}$$

Clearly, $f_{\rm out}$ is not analytic at the horizon $x_{\rm EH}$, but we can extend it by analytic continuation to the inside of the black hole through the lower half complex x-plane as [7]

$$(x-x_{\rm EH})\to|x-x_{\rm EH}|e^{-i\pi} = (x_{\rm EH}-x)^{-i\pi}\,, \tag{31}$$

$$f_{\rm out}\to\tilde{f}_{\rm out} = e^{-i\omega v_*}e^{2i\omega x_*}e^{2\pi\omega}\,. \tag{32}$$

The relative scattering probability of the outgoing wave at the horizon is

$$\left|\frac{f_{\rm out}}{\tilde{f}_{\rm out}}\right|^2 = e^{-4\pi\omega}\,. \tag{33}$$

Following Damour and Ruffini [7] and Sannan [8], it is easy to get the radiation spectrum of the particles from the black hole,

$$N_\omega = \frac{1}{e^{\omega/T_{v_*}}-1}\,, \tag{34}$$

$$T_{v_*} = \frac{1}{4\pi}\,. \tag{35}$$

From eq. (30), ω in eq. (34) is the radiation frequency at time coordinate v_*. Assuming the radiation frequency is $\tilde{\omega}$ at time coordinate v, then

$$\omega v_* = \tilde{\omega} v \,. \tag{36}$$

The solution of eq. (36) is

$$\omega = \tilde{\omega}/\kappa \,, \qquad \kappa = \frac{1}{v}\int A(x_{\mathrm{EH}}, v)\, \mathrm{d}v \,. \tag{37}$$

The distribution function of the radiation intensity is

$$N_{\tilde{\omega}} = \frac{1}{e^{\tilde{\omega}/T_v} - 1} \,. \tag{38}$$

The radiation temperature is

$$T_v = \frac{1}{4\pi v}\int A(x_{\mathrm{EH}}, v)\, \mathrm{d}v \,. \tag{39}$$

From eq. (35) and eq. (39) we know that at different time coordinate, temperature is different. At time coordinate v_*, the radiation temperature is a constant $1/4\pi$, it is not relative to the metric. At time coordinate v, the Hawking radiation temperature is relative not only to the metric but to the change of metric. From eq. (39) we think the Hawking radiation temperature is a compensate effect under the time scale transformation.

So we prove that at time coordinate v_*, through the generalized tortoise-type coordinate transformation, the KG equation in non-static plane-symmetric space-time and non-static spherically symmetric space-time will be changed into the standard wave motion equation which is near the horizon:

$$\frac{\partial^2 \psi}{\partial x_*^2} + 2\frac{\partial^2 \psi}{\partial x_* \partial v_*} = 0 \,, \tag{40}$$

$$\frac{\partial^2 \varrho}{\partial r_*^2} + 2\frac{\partial^2 \varrho}{\partial r_* \partial v_*} = 0 \,. \tag{41}$$

The equation deciding the location of the event horizon is

$$g_{00} + 2g_{01}\dot{x} = 0 \,, \tag{42}$$

the radiation temperature is

$$T_v = \frac{1}{4\pi v}\int A(x_{\mathrm{EH}}, v)\, \mathrm{d}v \,, \tag{43}$$

where

$$A(x_{\mathrm{EH}}, v) = -\left[\frac{g'_{00} + 2\,\dot{x}_{\mathrm{EH}}\, g'_{01}}{g_{01}} \right]_{x = x_{\mathrm{EH}}} . \tag{44}$$

In this article, the difficulty of calculating energy-momentum tensors is avoided, the Hawking radiation temperature and the location of the event horizon are obtained at non-static plane-symmetric space-time and non-static spherically symmetric space-time. We supply a dependable and direct new way for studying the Hawking quantum thermal effect.

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