The explicit expressions for the phase shifts of N-soliton solution of the KdV equation

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Summary. — The collisions among pure N solitons of the KdV equation are described by the solitons' phase shifts, which are dependent on the scattering data of the reflectionless potential $-N(N+1)\,\mathrm{sech}^2 x$. In the present work the scattering data in the case of arbitrary positive integer N are evaluated by direct scattering and the explicit expressions for the solitons' phase shifts are obtained according to the Inverse Scattering Transformation (IST).

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1. – Introduction

It is well known that the solution of the KdV equation

$$U_t - 6UU_x + U_{xxx} = 0 \tag{1}$$

with the reflectionless initial condition $-N(N+1)\,\mathrm{sech}^2 x$ $(N = 1, 2, 3, \ldots)$ will split into N solitons in the limit of $|t| \to \infty$. This fact shows that the N solitons are stable and resume both their form and velocity after collisions. These collisions can be analyzed by the phase shifts of the N solitons so that we may identify each soliton in the particle sense. However, the expressions for the phase shifts of N solitons are dependent on the reflectionless potential scattering data which are still undetermined with arbitrary positive integer N of the reflectionless potential parameter (*i.e.* the number of solitons) [1, 2]. In this work we will first get the reflectionless potential scattering data in the N-dependence form by solving the associated direct scattering equation, then evaluate the explicit expressions for the phase shifts of the N solitons by means of the IST method, and finally give an analytical function of (x, t) for the 3-soliton interactions as an application of the explicit expressions for the phase shifts of the N solitons.

2. – The reflectionless potential scattering data

Under the variable substitution [3] of $y = \tanh x$, the direct scattering equation with the reflectionless potential

$$-\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} - N(N+1)\,\mathrm{sech}^2 x\Psi = \lambda\Psi \tag{2}$$

can be transformed into the following:

$$(1-y^2)\frac{\mathrm{d}^2\Psi}{\mathrm{d}y^2} - 2y\frac{\mathrm{d}\Psi}{\mathrm{d}y} + \left[N(N+1) + \frac{\lambda}{1-y^2}\right]\Psi = 0\,, \tag{3}$$

where λ and Ψ are the eigenvalue and eigenfunction, respectively. Equation (3) is the well-known associated-Legendre equation. Therefore, the bound eigenvalues and the orthogonal normalized eigenfunctions will be as follows [3]:

$$\lambda_n = -n^2 \qquad (n = 1, 2, \ldots, N)\,, \tag{4}$$

$$\Psi_n = \sqrt{\frac{(N-n)!\,n}{(N+n)!}}\,\frac{\mathrm{sech}^n x}{N!\,2^n}\,\frac{\mathrm{d}^{N+n}}{\mathrm{d}y^{N+n}}(y^2-1)^N\,. \tag{5}$$

The $N+n$ order differentiation factor in eq. (5) can be written as

$$\frac{\mathrm{d}^{N+n}}{\mathrm{d}y^{N+n}}(y^2-1) = \frac{\mathrm{d}^{N+n}}{\mathrm{d}y^{N+n}}[(y+1)^N(y-1)]^N = \sum_{i=0}^{N+n}\frac{(N+n)!}{(N+n-i)!\,i!}[(y+1)^N]^{(i)}[(y-1)^N]^{(N+n-i)}\,. \tag{6}$$

It is obvious that only the terms of $n \leqslant i \leqslant N$ will remain after the evaluation of the differentiation in eq. (6). Then we have

$$\frac{\mathrm{d}^{N+n}}{\mathrm{d}y^{N+n}}(y^2-1)^N = \sum_{i=n}^{N}\frac{(N+n)!\,(N!)^2}{(N+n-i)!\,i!(N-i)!(i-n)!}(y+1)^{N-i}(y-1)^{i-n}\,. \tag{7}$$

The substitution of eq. (7) into eq. (5) with $y = \tanh x$ will yield

$$\Psi_n(x) = \frac{\sqrt{(N+n)!\,(N-n)!\,n}\,N!}{2^N}\,\mathrm{sech}^n x\sum_{i=n}^{N}\frac{(\tanh x+1)^{N-i}(\tanh x-1)^{i-n}}{(N+n-i)!\,(N-i)!\,(i-n)\,i!}\,. \tag{8}$$

As $z \to \infty$ ($\tanh x \to 1$), the summation in eq. (8) remains one term of $i = n$ only. So that both N- and n-dependent reflectionless potential scattering data will be

$$k_n = \sqrt{|\lambda_n|} = n \qquad (n = 1, 2, \ldots, N)\,, \tag{9}$$

$$C_n = \lim_{x\to\infty}\Psi_n e^{k_n x} = \frac{1}{n!}\sqrt{\frac{(N+n)!\,n}{(N-n)!}}\,. \tag{10}$$

3. – The explicit expressions for the phase shifts

Following the steps of the IST method [1, 2] with subsequent calculations, we have the asymptotic form of the N-soliton solution of the KdV equation with the reflectionless initial condition as follows:

(11) $$U(x, t) \xrightarrow[\xi_n \text{ fixed}]{t \to \pm\infty} -2 \sum_{i=1}^{N} k_n^2 \operatorname{sech}(k_n \xi_n + \delta_n^{\pm}),$$

where

(12) $$\xi_n = x - 4k_n^2 t,$$

(13) $$\delta_n^+ = \mathrm{Ln}\left[\frac{C_n}{\sqrt{2k_n}} \prod_{i=1}^{n-1} \frac{k_n - k_i}{k_n + k_i}\right],$$

(14) $$\delta_n^- = \mathrm{Ln}\left[\frac{C_n}{\sqrt{2k_n}} \prod_{i=n+1}^{N} \frac{k_i - k_n}{k_i + k_n}\right].$$

The n-dependence constants δ_n^+ and δ_n^- were defined as the phase shifts [1, 2] of the N solitons after their collisions. After the substitution of the reflectionless potential scattering data, eqs. (9), (10), into eq. (13) and eq. (14) with some algebraical calculations, we can finally obtain the explicit expressions for the phase shifts of the N solitons of the KdV equation as follows:

(15) $$\delta_n^+ = \mathrm{Ln}\left[\frac{(n-1)!}{(2n-1)!}\sqrt{\frac{(N+n)!}{2(N-n)!}}\right],$$

(16) $$\delta_n^- = \mathrm{Ln}\left[\frac{(2n)!}{(n)!}\sqrt{\frac{(N-n)!}{2(N+n)!}}\right],$$

with the special case of

(17) $$\delta_N^+ = \mathrm{Ln}\left[\frac{(N-1)!}{(2N-1)!}\sqrt{\frac{(2N)!}{2}}\right] \leqslant 0,$$

(18) $$\delta_N^- = \mathrm{Ln}\left[\frac{1}{N!}\sqrt{\frac{(2N)!}{2}}\right] \geqslant 0.$$

The sign of equality corresponds to the special case of the soliton number $N = 1$. It means that there exists no collisions for only one soliton, so the phase shift of one soliton is zero.

From eqs. (15), (16), the relative phase shifts of the N solitons can be calculated as

(19) $$\delta_n = \delta_n^+ - \delta_n^- = \mathrm{Ln}\left[\frac{(N+n)!\,(N-n)!\,n!}{(2n)!\,(2n-1)!\,(N-n)!}\right],$$

with which the conservation of the total phase shifts can be verified directly as

$$\sum_{n=1}^{N} \delta_n = \mathrm{Ln}\left[\prod_{n=1}^{N} \frac{(N+n)!\,(N-1)!\,n!}{(2n)!\,(2n-1)!\,(N-n)!}\right] = 0,$$

or, equivalently,

$$\sum_{n=1}^{N} \delta_n^+ = \sum_{n=1}^{N} \delta_n^- . \tag{21}$$

It is worthy to note that eq. (21) is true only in the asymptotic case. However, since the solution of the KdV equation splits into solitons rapidly as $|t| \to \infty$, the conservation of the total phase shifts may be a useful notion in the study of the KdV equation.

4. – The analytical formula of 3-soliton interactions

For $N = 3$, as an application of both our N- and n-dependent expressions of the phase shifts, eqs. (15), (16) and the scattering data, eqs. (9), (10), using the IST method with some tedious calculations we can get the analytical apparent function of (x, t) form for 3-soliton interactions of the KdV equation as

$$\begin{aligned} U(x, t) = -24[&\cosh(10x - 280t) + 10\cosh(8x - 216t) + 15\cosh(6x - 72t) + \\ &+ 30\cosh(6x - 216t) + 80\cosh(4x - 64t) + 50\cosh(2x - 8t) + \\ &+ 135\cosh(2x - 56t) + 25\cosh(2x - 152t) + 40\cosh(2x - 208t) + \\ &+ 126]/[\cosh(6x - 144) + 6\cosh(4x - 136t) + 15\cosh(2x - 80t) + 10\cosh 72t]^2 , \end{aligned} \tag{22}$$

which reduce exactly to $-12\,\mathrm{sech}^2 x$ at $t = 0$.

The corresponding asymptotic form of eq. (22) is

$$\begin{aligned} U(x, t) \xrightarrow[\xi_n \text{ fixed}]{\tau \to \pm\infty} -2\,\mathrm{sech}^2\left(\xi_1 \pm \frac{1}{2}\ln 6\right) - 8\,\mathrm{sech}^2\left(2\xi_2 \pm \frac{1}{2}\ln\frac{5}{3}\right) - \\ -18\,\mathrm{sech}^2\left(3\xi_3 \mp \frac{1}{2}\ln 10\right), \end{aligned} \tag{23}$$

where

$$\xi_n = x - 4n^2 t \qquad (n = 1, 2, 3). \tag{24}$$

Obviously, the analytical formula of 3-soliton interactions is much longer than that of the well-known 2-soliton interactions in ref. [1, 2]. From this we see the complexity of nonlinear interactions and the difference between nonlinear and linear interactions.

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