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NOTE BREVI

On the conservation theorems in classical field theory

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Summary. — The necessary and sufficient conditions for the validity of the classical conservation laws are explicitly identified.

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1. – The customary deductions of the classical (non-quantal) conservation theorems make use of invariance criteria—which restrict the form of the Lagrangian function—and of the equations of motion [1]. From the logical point of view this is a redundant procedure, since the conservation laws are clearly implied by the equations of motion, which yield the most detailed description of the system: indeed, in general the latter are *stronger* than the former. On the other hand, it is mathematically evident from the above deductions that the equations of motion are only *sufficient*, but *not* necessary conditions for the validity of the conservation laws. We wonder whether there exists a way to derive the conservation theorems, which demonstrates that the equations of motion are, roughly speaking, only "partially involved" in the various conservations: of energy-momentum, four-dimensional angular momentum, isospin, etc. Actually, there are in general many motions with the same values of the conserved quantities.

In the present paper we answer affirmatively to the above question. We limit ourselves for brevity to the case of the classical field theory in Minkowskian space-time, from which one can proceed, with suitable changes, to the consideration of the systems endowed with only a finite number of degrees of freedom. We show in the appendix that our treatment can be extended to the Einsteinian theory of the non-symmetric field [2].

2. – As it is well known, in the general theory of relativity the conservation theorems have a very peculiar status [3]. But if we reformulate the special relativity in general co-ordinates (see, *e.g.*, Fock [1]), some analytical procedures of Einstein's theory of gravitation can be advantageously used for our purpose.

We start from a closed system of coupled classical fields moving in Minkowskian space-time, which we describe with the following real pseudometric: $ds^2 =$

$\eta_{mn}\,\mathrm{d}y^m\,\mathrm{d}y^n$, $\eta_{mn} = 0$ for $m \neq n$, $\eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1$. In a general frame $x^m = x^m(y)$ we have, of course, $\mathrm{d}s^2 = \gamma_{mn}(x)\,\mathrm{d}x^m\,\mathrm{d}x^n$ with a γ_{mn} such that the Riemann curvature tensor remains identically equal to zero. Let us consider the *invariant* action integral

$$A = \int_G \mathfrak{L}[u_a(x),\ \partial_l u_a(x),\ \gamma_{mn}(x)](\mathrm{d}x)\,, \tag{2.1}$$

where $\mathfrak{L}$ is the Lagrangian scalar density (which does not depend explicitly on x); the $u_a(x)$'s ($a = 1, 2, \ldots, f$) are linearly independent field functions (the index a specifies both a field and a component of a given field); $\partial_l := \partial/\partial x^l$; $(\mathrm{d}x) := \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3$; G is an arbitrary space-time domain. The fundamental tensor γ_{mn} is here considered as the representative of a "non-rigid" pseudometric, and therefore it has been formally promoted to the rank of a field function.

We perform now an infinitesimal change of frame:

$$\delta x^j := x^j - x'^j = \xi^j(x')\,, \tag{2.2}$$

where

$$\xi^j(x') := \varepsilon^j \chi(x')\,; \tag{2.3}$$

$\chi(x')$ is an arbitrary function *vanishing at the boundary* ∂G of G, and ε^j is an arbitrary infinitesimal constant four-vector. The transformation (2.2), (2.3) induces the following variations $\delta\gamma_{mn}$ and δu_a of the functional forms of γ_{mn} and u_a:

$$\delta\gamma_{mn}(x) := \gamma'_{mn}(x) - \gamma_{mn}(x) = \gamma_{jn}(x)\,\partial_m \xi^j(x) + \\ + \gamma_{mj}(x)\,\partial_n \xi^j(x) + \xi^j(x)\,\partial_j \gamma_{mn}(x)\,; \tag{2.4}$$

$$\delta u_a(x) := u'_a(x) - u_a(x) = \varepsilon^j v_{aj}(x) \equiv \xi^{j'}(x)\,w_{aj}(x)\,, \tag{2.5}$$

where the v_{aj}'s and the w_{aj}'s are functions which depend on the tensor or spinor properties of the u_a's. Putting

$$\mathfrak{L}^a := \frac{\partial\mathfrak{L}}{\partial u_a} - \partial_l\left[\frac{\partial\mathfrak{L}}{\partial(\partial_l u_a)}\right], \tag{2.6}$$

$$\mathfrak{L}^{mn} := \frac{\partial\mathfrak{L}}{\partial\gamma_{mn}} - \partial_l\left[\frac{\partial\mathfrak{L}}{\partial(\partial_l \gamma_{mn})}\right], \tag{2.7}$$

one obtains, with the usual partial integrations:

$$\delta A = \int_G (\mathfrak{L}^a\,\delta u_a + \mathfrak{L}^{mn}\,\delta\gamma_{mn})(\mathrm{d}x) = 0\,. \tag{2.8}$$

By substituting in (2.8) the formulae (2.4), (2.5), and by performing some simple computations (see, *e.g.*, Schrödinger [3]), we have finally

$$\int_G \mathfrak{L}^a w_{aj}\,\xi^j(\mathrm{d}x) = 2\int_G (D_m \mathfrak{L}^m{}_j)\,\xi^j(\mathrm{d}x)\,, \tag{2.9}$$

where $D_m \mathfrak{L}^m{}_j := \partial_m \mathfrak{L}^m{}_j - \mathfrak{L}^m{}_j \Gamma^l{}_{mj}$ is the covariant derivative of $\mathfrak{L}^m{}_j$. It follow immediately that

$$\mathfrak{L}^a w_{aj} = 2 D_m \mathfrak{L}^m{}_j , \tag{2.10}$$

i.e. that *necessary and sufficient* conditions for the validity of the energy-momentum differential conservation law $D_m \mathfrak{L}^{mj} = 0$ are the four relations $\mathfrak{L}^a w_{aj} = 0$, which are manifestly *weaker* than the equations of motion $\mathfrak{L}^a = 0$ ($a = 1, 2, \ldots, f$). The present treatment assures automatically the *symmetry* of the energy-momentum tensor: $\mathfrak{L}^{mn} = \mathfrak{L}^{nm}$. The passage from the differential form $D_m \mathfrak{L}^{mj} = 0$ to the integral form of the energy-momentum conservation law does not present any conceptual difficulty, by virtue of the fact that the Riemann curvature tensor is identically equal to zero. The Minkowskian form, η_{mn}, of the fundamental tensor is here only a particular form of the "non-rigid" pseudometric. Adopting Minkowskian co-ordinate y^m, the above differential law can be simply written $\partial_m \mathfrak{L}^{mj} = 0$. Owing to the symmetry of $\mathfrak{L}^{mn}$, the conservation law of the total four-dimensional angular momentum ("orbital" term plus polarization-spin term) $y^n \mathfrak{L}^{jm} - y^m \mathfrak{L}^{jn}$ can be immediately obtained.

3. – Finally, we wish to derive the isospin conservation law. Consider in the space of the isospin field functions the infinitesimal transformations $\delta u_a(y) := u'_a(y) - u_a(y)$ which are ideally associated with the infinitesimal rotations

$$\delta z^\mu := z'^\mu - z^\mu = z_\nu \varepsilon^{\mu\nu} \qquad (\mu, \nu = 1, 2, 3), \tag{3.1}$$

—where the $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$ are infinitesimal constants—in that fictive three-dimensional Euclidean space which is commonly callled "isospin space". With a compact and evident notation, we can write

$$\delta u_a(y) = \frac{1}{2} R_{a\mu\nu}(y)\, \varepsilon^{\mu\nu}; \tag{3.2}$$

for conceptual simplicity we have here employed Minkowskian co-ordinates y^j.

If the Lagrangian $\mathfrak{L}$ is *invariant* under the variations (3.2), we have—bearing in mind that $\delta\partial_l = \partial_l \delta$—:

$$\mathfrak{L}^a R_{a\mu\nu}(y)\, \varepsilon^{\mu\nu} + \partial_l \left[\frac{\partial \mathfrak{L}}{\partial(\partial_l u_a)} R_{a\mu\nu}(y) \right] \varepsilon^{\mu\nu} = 0 , \tag{3.3}$$

from which

$$\mathfrak{L}^a R_{a\mu\nu} = -\partial_l \left[\frac{\partial \mathfrak{L}}{\partial(\partial_l u_a)} R_{a\mu\nu} \right]; \tag{3.4}$$

consequently, necessary and sufficient conditions for the conservation law of isospin tensor density

$$\mathfrak{J}^l{}_{\mu\nu} := \frac{\partial \mathfrak{L}}{\partial(\partial_l u_a)} R_{a\mu\nu} \tag{3.5}$$

are the three equations

$$\mathfrak{L}^a R_{a\mu\nu} = 0 \, .$$

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APPENDIX

In Einstein's *Relativistic Theory of the Non-symmetric Field* [2] the conservation theorem of energy and momentum has a status formally similar to that illustrated in the above sections concerning the Minkowskian field theories. Let us summarize the last Einsteinian creation. The field equations are:

$$S_{ik} = 0 \, ; \qquad \mathfrak{N}^{ik}{}_l = 0 \, , \tag{A.1}$$

where

$$S_{ik} := \partial_s U^s{}_{ik} - U^s{}_{it} U^t{}_{sk} + \frac{1}{3} U^s{}_{is} U^t{}_{tk} \tag{A.2}$$

is the contracted curvature tensor and $U^l{}_{ik}$ is a pseudo-connection thus defined in terms of the basic non-symmetric connection Γ^l_{ik}:

$$U^l{}_{ik} := \Gamma^l_{ik} - \Gamma^t{}_{it} \delta^l_k , \tag{A.3}$$

while $\mathfrak{N}^{ik}{}_l$ is a kind of covariant derivative of $\mathfrak{g}^{ik}$, given by

$$\mathfrak{N}^{ik}{}_l := \partial_l \mathfrak{g}^{ik} + \mathfrak{g}^{sk}\left(U^i{}_{sl} - \frac{1}{3} U^t{}_{st} \delta^i_l\right) + \mathfrak{g}^{is}\left(U^k{}_{ls} - \frac{1}{3} U^t{}_{ts} \delta^k_l\right), \tag{A.4}$$

and $\mathfrak{g}^{ik}$ is a contravariant tensor density. Einstein introduces the following Lagrangian scalar density:

$$\mathfrak{H} := \mathfrak{g}^{ik} S_{ik} ; \tag{A.5}$$

by variation of $\mathfrak{H}$ with respect to $\mathfrak{g}$ and U—which are the keystone of Einstein's unusual geometry—we obtain

$$\delta\mathfrak{H} = S_{ik} \delta\mathfrak{g}^{ik} - \mathfrak{N}^{ik}{}_l \delta U^l{}_{ik} + \partial_s(\mathfrak{g}^{ik} \delta U^s{}_{ik}) \, . \tag{A.6}$$

When these variations are the variations of the functional forms induced by an infinitesimal co-ordinate transformation

$$x'^i = x^i + \xi^i(x) \, , \tag{A.7}$$

we have obviously $\delta\mathfrak{H} = 0$, from which it follows $\partial_s(\mathfrak{g}^{ik} \delta U^s{}_{ik}) = 0$ *if* the equations of motion (A.1) are satisfied. For a ξ^i *independent* of x, we obtain

$$\delta U^s{}_{ik} = -\partial_t U^s{}_{ik} \xi^t , \tag{A.8}$$

and consequently

(A.9) $$\partial_s(\mathfrak{g}^{ik}\partial_t U^s{}_{ik}) = 0 .$$

Einstein interprets these equations as the differential conservation laws of energy and momentum. Now, we can remark that the above assumptions $S_{ik} = \mathfrak{N}^{ik}{}_l = 0$ are unduly *restrictive*: as a matter of fact, the necessary and sufficient conditions for the validity of (A.9) is

(A.10) $$S_{ik}\,\delta\mathfrak{g}^{ik} - \mathfrak{N}^{ik}{}_l\,\delta U^l{}_{ik} = -\,S_{ik}\,\xi^s\,\partial_s\,\mathfrak{g}^{ik} + \mathfrak{N}^{ik}{}_l\,\xi^s\,\partial_s\,U^l{}_{ik} = 0 ,$$

from which the four equations

(A.11) $$S_{ik}\,\partial_s\,\mathfrak{g}^{ik} = \mathfrak{N}^{ik}{}_l\,\partial_s\,U^l{}_{ik}$$

follow immediately. *Q.e.d.*

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[1] See, *e.g.*, NOETHER E., *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl.* (1918) 235; BELINFANTE F. J., *Physica*, **6** (1939) 887; HILL E. L., *Rev. Mod. Phys.*, **23** (1951) 253; KÄLLÉN G., *Quantenelektrodynamik*, Handb. d. Physik, Vol. V/1 (Springer-Verlag, Berlin) 1958, sects. 3 and 4; FOCK V., *The Theory of Space, Time and Gravitation*, 2nd revised edition (Pergamon Press, Oxford) 1964, sects. 48 and 49; JOST R., *The General Theory of Quantized Fields* (American Mathematical Society, Providence, R.I.) 1965, p. 27 and following; STEUDEL H., *Ann. Phys.* (*Leipzig*), **20** (1967) 110; GUPTA S. N., *Quantum Electrodynamics* (Gordon and Breach Science Publishers, New York) 1977, sect. 8; BOGOLIUBOV V. N. and SHIRKOV D. V., *Introduction to the Theory of Quantized Fields*, 3rd edition (Wiley and Sons, New York) 1980, sect. 2.

[2] EINSTEIN A., *The Meaning of Relativity*, 5th edition (Princeton University Press, Princeton, N.J.) 1955, Appendix II, and in particular p. 163.

[3] See, *e.g.*, SCHRÖDINGER E., *Space-Time Structure* (Cambridge University Press, Cambridge) 1960, p. 93 and following; FOCK V., *op. cit.* in [1], sects. 88, 89, 90, 91; WEYL H., *Raum-Zeit-Materie*, Siebente Auflage (Springer-Verlag, Berlin) 1988, sects. 30 and 41.